Math 295: Introduction to Proof-Oriented Mathematics

1. Course Number: Math 295

2. Title: Introduction to Proof-Oriented Mathematics.

3. Credits: 4

4. Term, place, time and instructor: Spring term, preferably Deady or Fenton halls, TBD, Vaintrob in spring 2008.

Faculty who have taught this course as a special course (labeled MA 199) are Sinha and Polishchuk. Faculty who have expressed an interest in teaching this course in the future are Dugger, Levin, Bownik, Brundan, Libeskind. But note that any of our faculty could teach the course.

5. Position in the curriculum: We propose that this satisfy the science group requirement... (more later). The primary set of fundamental questions addressed by this course include:

- What is a proof? How are basic proofs constructed?
- What different kinds of proofs are there? How does one understand and use logical constructions such as converse and contrapositive? What is induction?
- What does it mean to prove some fact which is already familiar and believable?
- How does one start with some interesting problem, work through some examples, and then attempt to capture the essence of what is happening with a proof?

6. Format: This course is nominally a lecture course, but in some ways it is perhaps closer to a laboratory course. Topics are covered in two-day blocks. Before the first class of each block, students are assigned some problems (typically 5-12 of them) with a wide range of difficulty levels. Every student turns in work on a few problems, with generous credit being given for working out relevant examples and other forms of progress since they are trying these problems entirely on their own. Complete solutions are generally not expected. Some of these problems are then the focus of discussion and further work during the first class, mostly in small groups. The second class in each block is more of a lecture, introducing relevant formalism and then giving proofs in the context of of careful solutions of problems. This lecture is close to what a "normal" lecture would be, but should have a different feel for students, colored by the previous experience working with these ideas without the formalism. Students see proofs in lecture as capturing what they had seen (at least in part) on their own, rather than an unmotivated formality. At the end of each week, students then go back to the problems previously assigned and turn in more polished solutions of some of those problems using ideas from the lectures.

7: Outline of subject and topics covered: The philosophy is to introduce students to proofs and formalism in the context they naturally arise for working mathematicians, namely while trying to understand interesting problems. Proofs are the language in which modern mathematicians establish truth and communicate ideas. They are virtually absent from the high-school curriculum, and among our own calculus courses for example are only prevalent in the Honors Sequence. Many of our majors have difficulty transitioning to doing proofs in our 300 and 400 level courses, so this course is designed to familiarize them with this basic technique.

This class is organized around the idea of having students work on interesting problems on their own before seeing polished lectures which address those problems, as outlined in the discussion of Format above. A central desired outcome is for students to have two realizations. One is the philosophical lesson that the fine-tuned mathematics they see in lectures wasn't "born" that way, and the other is the practical lesson that to really understand some mathematics they need to gain some hands-on experience, even when a topic isn't presented in a hands-on way.

The mathematical topics covered lend themselves well to doing elementary proofs. Some of the topics explored include:

- What are equations (to a mathematician)? What are sets?
- What is the relationship between set theoretic and logical properties, for example between inclusion of sets and logical implication? How does the existence of the empty set bear on the truth of mathematical propositions?
- How can induction be used in a wide array of settings, including establishing equalities and inequalities and analyzing "games"?
- How do we count basic structures such as subsets or permutations?
- How do we establish basic properties of numbers such as unique factorization of integers into primes, or ability to divide with remainder?
- What is Fermat's Little Theorem and how does it relate to cryptography?
- How do graphs and associated structures (orientations and colorings) model phenomena, and how may we use all of the tools at our disposal to analyze them?

8. Course materials: The class has been using the book "Mathematical thinking: problem solving and proofs" by D'Angelo and West. This text was chosen because it has by far the best collection of problems of any textbook we found which aims to introduce proofs. We typically cover roughly seven chapters such as 1-3, 5-7 and 11. At times the book is challenging for students to read because of its formal nature, but reading the book together as part of group work is one of the great benefits of class. A supplementary text which introduces some topics with a more elementary exposition is “The heart of mathematics” by Burger and Starbird. Six copies of this text were purchased and donated so they are available to students from the shelves of the undergraduate math lounge (commonly known as “Hilbert Space.”)

9. Instructor expectations of students: In each week the schedule is as follows.

- Monday, work on problems in groups, building on work done before class.
- Tuesday, lecture on material relevant to Monday’s problems.
- Wednesday, work on a new set of problems in groups, as on Monday.
- Friday, lecture on material relevant to Wednesday’s problems.

Students are typically assigned many more problems than they would do. In a typical week they are asked to attempt 4-5 problems before each Monday or Wednesday class. Then they re-do about five problems (usually with some overlap with the ones they did) for their more formal write-ups.

The topics covered week-by-week have typically been as follows.

1. Introduction to class. Arithmetic and geometric mean. (Chapter 1, pages 1-12 of “Mathematical Thinking”) Problems assigned 1.4, 1.6, 1.8, 1.10, 1.11, 1.24, 1.35.
2. Logic, including quantifiers and deduction. (Chapter 2, pages 25-31 and 35-43) Problems assigned M: 2.2, 2.4. 2.10, 2.11, 2.18, 2.22, 2.32, 2.33, 2.34, 2.42; W: 2.16, 2.30, 2.31, 2.40*, 2.51*.
3. Mathematical induction. (Chapter 3, pages 50-62) Problems assigned M: 3.1, 3.5, 3.7, 3.11, 3.29, 3.34, 3.58, 3.59; W: 3.8, 3.23, 3.39, 3.38, 3.49*, 3.58, 3.59, 3.62, 3.65.
4. Divisibility and factorization. (Chapter 4(a) pages76-80 and begin Chapter 6 pages 123-125) Problems assigned M: 4.2-8; W: 4.15*, 4.13*, 4.16, 6.2, 6.5, 6.6*, 6.8, 6.15*.
5. The Euclidean algorithm. (Chapter 6 pages 126-130) Problems assigned M: 6.17, 6.22, 6.23, 6.24, 6.34, 6.44, 6.46, 6.5 plus review for exam.
6. Midterm exam. Modular arithmetic. (Chapter 7 pages 139-146) Problems assigned M: 6.18, 6.25, 6.36, 6.54, 7.2, 7.3, 7.4, 7.23, 7.28 W: 7.5, 7.8, 7.9, 7.17, 7.19, 7.25, 7.42

7. Fermat's little theorem. Introductory cryptography. (Chapter 7 pages 147-149 and Chapter 2.5 of "Heart of Mathematics") Problems assigned M: 7.1, 7.10, 7.11, 7.16, 7.30, 7.33, 7.43 W: 4.20, 4.21, 4.22, 4.25, 4.30, 4.34, 4.42, 4.48, 4.49, 4.51

8. Combinatorics and bijective proofs. (Chapter 5 pages 100-110) Problems assigned M: 5.4, 5.5, 5.6, 5.9, 5.10, 5.11, 5.13, 5.18, 5.19, 5.21, 5.28, 5.37, 5.38, 5.39 W: 5.49, 5.50, 5.58 (start with [3]), 5.62, 5.64

9. Combinatorics and graph theory. (Chapter 5 pages 111-113 and Chapter 11 pages 202-206) Problems assigned W: 11.7, 11.8, 11.10, 11.11, 11.12, 11.16, 11.18, 11.19, 11.22.

10. Graph theory and a bit of topology. (Chapter 11 pages 207-215 and 219-228) Problems assigned Tu: 11.26, 11.27, 11.28, 11.31, 11.38, 11.39, 11.40, 11.4. Prepare for the final exam.

10. Assessment: For each set of problems, students are graded for their work of three different types - on their own before a topic is covered in class, in groups during class, and then more polished solutions of a few problems due at the beginning of the following week. The first two passes at the problems are looked at by the instructor and graded generously. Students need not get full solutions, but are expected to try relevant examples and make some attempts at partial proofs. The polished solutions, due after students have been shown many solutions of similar problems, will be marked and graded both for mathematical correctness and for exposition.

There is a mid-term and a final, which emphasize familiarity with important proofs rather than cleverness within time constraints. The grading will be roughly as follows.

- Class participation ("first" days, mostly): 15%
- Rough homework: 11%
- Correctness of polished homework: 12%
- Exposition in polished homework: 12%
- Mid-term: 20%; Final: 30%

Student workload expectations: Students should spend 2-3 hours before the Monday and Wednesday classes working on problems on their own. They should spend 3 hours over the weekend writing up the polished solutions of the few reassigned problems from the previous week. There will be four class hours each week, giving a total of roughly twelve hours per week which students are expected to spend on the class.