INTRODUCTION TO GOODWILLIE CALCULUS

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1. Introduction

These are notes from a course taught in Spring quarter 2009 on Goodwillie's calculus of functors. Our primary reference is [3].

The general idea is as follows. We wish to study some functor $F : \mathcal{C} \to \mathcal{D}$ in terms of functors that have simpler behavior in some homotopy theoretic way. In this way, we hope to be able to say things about $F(X)$ for some object X of $\mathcal{C}$ that would be hard to work out directly. For easy of exposition and because this is the example we are most interested in, we will take $\mathcal{C}$ to be the category of base-pointed topological spaces much of the time (which we'll denote $\underline{\mathbf{Spaces}}_*$), and we will often take $\mathcal{D}$ to be the same category. In fact, what one really needs is for both $\mathcal{C}$ and $\mathcal{D}$ to be model categories, but we won't work in this much generality.

Goodwillie frames his constructions analogously with the study of Taylor polynomials and Taylor series in functions of a real variable. The analogy proceeds as follows:

Calculus of a real variable	**Calculus of functors**
$f : \mathbf{R} \to \mathbf{R}$ a function of a real variable.	$F : \mathcal{C} \to \mathcal{D}$ a functor between two appropriate categories.
Polynomial functions of degree n.	n-excisive functors $\mathcal{C} \to \mathcal{D}$.
$P_{n,a}(f) : \mathbf{R} \to \mathbf{R}$ the nth Taylor polynomial of the function f.	$P_nF : \mathcal{C} \to \mathcal{D}$ an n-excisive approximation to F.
The kth derivative at a of the function f. The collection of these for $0 \leq k \leq n$ determine $P_{n,a}f$.	The kth derivative of the functor F. The collection of these for $0 \leq k \leq n$ together with extension data determins the functor P_nF.
Analytic functions. So $f(x)$ is determined by the $P_{n,a}(f)(x)$ in a neighborhood of a.	Suitably analytic functors. So $F(X)$ is determined by $(P_nF)(X)$ if F is analytic at X.

These notes are in three sections.

1. First we describe the definitions and constructions required to make P_nF from F (and the corresponding natural transformation and related functors).
2. Then we apply these ideas to deduce the James splitting of $\Sigma^\infty \Omega \Sigma X$ and more generally the Snaith splittings of $\Sigma^\infty \Omega^n \Sigma^n X$ in an easy way.

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(3) We describe the approach to chromatic homotopy theory uncovered by Arone and Mahowald by studying the Taylor tower of the identity functor [1].

The goal is thus to describe this chunk of machinery, show how this point of view can be used to reprove some classical results, and then show how it leads to some new insights.

2. CONSTRUCTING $P_n F$

Both the construction of the approximations $P_n F$ and the definition of n-excisive depend on the notion of homotopy colimits. The definition of n-excisive also depends on the notion of homotopy limits.

2.1. **Homotopy colimits.** In preparation for discussion of homotopy colimits we recall the ordinary colimit construction in $\underline{\mathbf{Spaces}}_*$.

2.1.1. *Colimits.* Consider some diagram of spaces

$$\begin{tikzcd}
\cdots \arrow[r] & X_\alpha \arrow[r, "g_{\alpha\beta}"] \arrow[dr, "g_{\alpha\gamma}"] & X_\beta \arrow[r] & \cdots \\
& \cdots \arrow[r] & X_\gamma \arrow[r] & \cdots
\end{tikzcd}$$

The colimit of this diagram, $\operatorname{colim}_\alpha X_\alpha$ is an object X together with maps $i_\alpha : X_\alpha \to X$ satisfying the following universal property:

> If for each α there is a map $f_\alpha : X_\alpha \to Y$ such that for each $g_{\alpha\beta}$ we have $f_\alpha = f_\beta \circ g_{\alpha\beta}$, then there is a unique map $f : \operatorname{colim}_\alpha X_\alpha \to Y$ such that $f \circ i_\alpha = f_\alpha$ for all α.

We want to reformlate this in a slightly more abstract categorical way. Let I be our index category (we require that it be small). Let

$$\underline{\mathbf{Spaces}}_*^{\,I} = \text{the collection of functors } F : I \to \underline{\mathbf{Spaces}}_*.$$

We'll consider this to be a category whose objects are these functors with morphism sets being natural transformations between these functors.

There is an obvious functor $Y \mapsto \underline{Y}$ from $\underline{\mathbf{Spaces}}_*$ to $\underline{\mathbf{Spaces}}_*^{\,I}$. Here Y is a space, and $\underline{Y}$ is the constant functor that sends each object of I to Y and each morphism of I to the identity map of Y.

The colimit is a functor the other way. That is

$$\operatorname{colim} : \underline{\mathbf{Spaces}}_*^{\,I} \to \underline{\mathbf{Spaces}}_*.$$

A construction for the colimit is as follows

$$\operatorname{colim} F = \bigvee_{\alpha \in \operatorname{ob} I} F(\alpha) / \sim$$

where for any $x \in F(\alpha)$, $m \in \operatorname{Hom}_I(\alpha, \beta)$, $x \sim F(m)(x)$. One uses the standard property of quotient spaces to check this satisfies the universal property given above.

In terms of the language we've just introduced, the universal property has a simpler statement:

$$\operatorname{Hom}_{\underline{\mathbf{Spaces}}_*^{\,I}}(G, \underline{Y}) = \operatorname{Hom}_{\underline{\mathbf{Spaces}}_*}(\operatorname{colim} G, Y).$$

In other words, the colimit functor is left adjoint to the functor taking a space to the constant diagram.

2.1.2. *Some Index Categories.*

Definition 1. We let $\underline{n}$ as usual denote the set $\{1, 2, \ldots, n\}$. We let $P(\underline{n})$ be the power set of $\underline{n}$ considered as a small category. The morphisms will be the inclusions of subsets.

So $P(\underline{2})$ can be represented as a square (neglecting both the identity morphism and the obvious compositions):

$$\begin{tikzcd}
\emptyset \arrow[r, "i_{\emptyset\{1\}}"] \arrow[d, "i_{\emptyset\{2\}}"] & \{1\} \arrow[d, "i_{\{1\}\{1,2\}}"] \\
\{2\} \arrow[r, "i_{\{2\}\{1,2\}}"] & \{1,2\}
\end{tikzcd}$$

Thus an element of $\underline{\mathbf{Spaces}}_*^{P(\underline{2})}$ is simply a commutative square.

Similarly $P(\underline{3})$ can be represented as a cube, and a $P(\underline{3})$ diagram is a commutative cube of spaces, $P(\underline{n})$ is an n-dimensional hypercube and a $P(\underline{n})$ diagram is an n-dimensional hypercube of spaces with all squares of maps commuting.

We'll need two more collections of index categories.

Definition 2. $P_0(\underline{n})$ is the subcategory of $P(\underline{n})$ which includes everything except the empty set. $P'(\underline{n})$ is the subcategory of $P(\underline{n})$ which includes everything except the set $\underline{n}$.

So $P'(\underline{2})$ is the square without the lower right hand corner. We'll follow the usual convention and if we have a functor $G : P'(\underline{2}) \to \underline{\mathbf{Spaces}}_*$, we'll often write X_S for $G(S)$ so that we represent that functor by the following diagram

$$\begin{tikzcd}
X_\emptyset \arrow[r, "i_1"] \arrow[d, "i_2"] & X_1 \\
X_2 &
\end{tikzcd}$$

Since the category is very small, this colimit can be represented in a very simple way.

$$\operatorname{colim} X_S = (X_1 \vee X_2)/\sim \quad = (X_1 \vee X_2 \vee X_\emptyset)/\simeq \tag{3}$$

where for each $x \in X_\emptyset$, $i_1(x) \sim i_2(x)$ and $x \simeq i_1(x) \simeq i_2(x)$ is the description of the two equivalence relations.

2.1.3. *Failure of homotopy invariance.* From the perspective of homotopy theory, the definition of colimit has a major drawback. If I take a diagram and replace some spaces with homotopy equivalent spaces, that may change the homotopy type of the colimit. The standard example is to compare

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \{*\} \\
\{*\} &
\end{tikzcd}$$

with

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & CX \\
CX &
\end{tikzcd}$$

(with both maps being inclusion into the open end of the cone). Using (3), it is easy to check that the colimit of the first diagram is $*$ and the colimit of the second diagram is ΣX.

2.1.4. *Homotopy colimits for $P'(\underline{2})$.* There are two things we'd like from a homotopy colimit.

1. We would like it if when we changed the diagram to a homotopic diagram we got the same homotopy colimit.
2. We want given maps $f_\alpha : X_\alpha \to Y$ such that for each $g_{\alpha\beta}$ we have $f_\alpha \simeq f_\beta \circ g_{\alpha\beta}$, then there is a unique (up to homotopy) map $f : \operatorname{hocolim}_\alpha X_\alpha \to Y$ such that $f \circ i_\alpha \simeq f_\alpha$ for all α.

It turns out that we can get the first condition, but not quite the second condition.

At this point I should define the homotopy colimit, but instead I'll defer the general definition and just give a construction that works for the special case $P'(\underline{2})$.

Definition 4. The *homotopy colimit* (also known as the *homotopy pushout*) of the diagram

$$\begin{tikzcd}
X_\emptyset \arrow[r, "i_1"] \arrow[d, "i_2"] & X_1 \\
X_2 &
\end{tikzcd}$$

is $(X_1 \vee X_2 \vee X_\emptyset \times I)/\sim$ where for each $x \in X_\emptyset$, $(x, 0) \sim i_1(x)$ and $(x, 1) \sim i_2(x)$.

We have the following.

Proposition 5. *Given $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$ such that $f_1 \circ i_1 \simeq f_2 \circ i_2$ there is a map $f : \operatorname{hocolim} X_S \to Y$ such that $f \circ i_{X_1} = f_1$ and $f \circ i_{X_2} = f_2$.*

The proof is straightforward. f is defined by f_1 on X_1, f_2 on X_2 and the homotopy from $f_1 \circ i_1$ to $f_2 \circ i_2$ on $X_\emptyset \times I$.

The map f fails to be unique, even up to homotopy. If there are two homotopies $f_1 \circ i_1$ to $f_2 \circ i_2$ which are not themselves homtopic, they will lead to two non-homotopic possibilities for f.

Exercise 1. Suppose i_1 and i_2 are both inclusions of sub-CW-complexes. Then

$$\operatorname{colim}\left(\begin{tikzcd}
X_\emptyset \arrow[r, "i_1"] \arrow[d, "i_2"] & X_1 \\
X_2 &
\end{tikzcd}\right) \simeq \operatorname{hocolim}\left(\begin{tikzcd}
X_\emptyset \arrow[r, "i_1"] \arrow[d, "i_2"] & X_1 \\
X_2 &
\end{tikzcd}\right)$$

2.2. **The P_0 and P_1 functors.** We will defer a serious discussion of P_0 (which won't be that serious anyhow), but for now we note that P_0F will be defined so that

$$(P_0F)(X) \simeq F(*).$$

So P_0 is (at least up to homotopy) the constant functor at $F(*)$. A word is in order about $F(*)$ perhaps. Many of the functors we are used to considering satisfy $F(*) \simeq *$, but of course this isn't automatic. For example, the functor $F(X) = X \times Y$ is a homotopy functor, but does not satisfy $F(*) = *$.

Definition 6. A functor F is a *homotopy functor* if $f : X \to Y$ a weak equivalence implies $F(f) : F(X) \to F(Y)$ is a weak equivalence.

Definition 7. A homotopy functor F is *reduced* if when X is weakly equivalent to a point, so is $F(X)$.

To define P_1F, we need the notion of excisive (or 1-excisive). This corresponds under our analogies to being polynomial of degree 1.

Definition 8. A functor F is *excisive* (or 1-*excisive*) if F takes homotopy pullback squares to homotopy pushout squares.

We haven't quite given enough information to understand this yet. A commutative square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd}$$

is a homotopy pushout square if W is weakly homotopy equivalent to the homotopy cofiber of the rest of the square by the obvious map.

Definition 9. The homotopy pullback of the diagram

$$\begin{tikzcd} & Y \arrow[d, "f"] \\ Z \arrow[r, "g"] & W \end{tikzcd}$$

is the space

$$\{(y, z, \omega) \in Y \times Z \times W^I : f(y) = \omega(0), g(z) = \omega(1)\}.$$

In other words, it consists of pairs of points, one in Y, one in Z and a path between their images in W. A commutative square

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "k"] & Y^f \arrow[d] \\ Z^g \arrow[r] & W \end{tikzcd}$$

is a homotopy pullback square if X is weakly homotopy equivalent to the homotopy pullback of the rest of the square by the map $x \mapsto (h(x), k(x), c_{f \circ h(x)})$.

Proposition 10. *If E is a spectrum representing a homology theory $E_*(-)$, then the functor*

$$F(X) = \Omega^\infty(\Sigma^\infty X \wedge E)$$

is excisive.

Proof. Let's suppose that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[d, "h"] \\ Z \arrow[r, "k"] & W \end{tikzcd}$$

is a homotopy pushout square. Then W is weakly equivalent to the homotopy pushout of the rest of the square - that is to to $[Y \vee Z \vee (X \times I)]/\sim$. We write $W = U \cup V$ where $U = Y \vee (X \times [0, 3/4))/\sim$ and $V = Z \vee (X \times (1/4, 1])/\sim$. Notice that Y is an s.d.r. of U and Z is an s.d.r. of V. Also, X is an s.d.r. of $U \cap V$ and $W = U \cup V$.

So if we apply Mayer-Vietoris for $E_*(-)$, we get a long exact sequence

$$\cdots \to E_k(X) \xrightarrow{(f_*, g_*)} E_k(Y) \oplus E_k(Z) \xrightarrow{(h_*, -k_*)} E_k(W) \xrightarrow{\delta} E_{k-1}(X) \to \cdots$$

Now since homology commutes with the functor Σ^∞, we also get the same long exact sequence from the square of spectra

$$\begin{tikzcd} \Sigma^\infty X \arrow[r, "f"] \arrow[d, "g"] & \Sigma^\infty Y \arrow[d, "h"] \\ \Sigma^\infty Z \arrow[r, "k"] & \Sigma^\infty W \end{tikzcd}$$

Since (for a spectrum A), $E_*(A) = \pi_*^{st}(A \wedge E)$, the square we get by smashing with E

$$\begin{tikzcd} \Sigma^\infty X \wedge E \arrow[r, "f"] \arrow[d, "g"] & \Sigma^\infty Y \wedge E \arrow[d, "h"] \\ \Sigma^\infty Z \wedge E \arrow[r, "k"] & \Sigma^\infty W \wedge E \end{tikzcd}$$

gives a long exact Mayer-Vietoris sequence in homtopy by using the naturality of the vertical isomorphisms:
(11)

$$\begin{tikzcd} E_k(X) \arrow[r, "(f_*, g_*)"] \arrow[d, "\approx"] & E_k(Y) \oplus E_k(Z) \arrow[r, "(h_*, -k_*)"] \arrow[d, "\approx"] & E_k(W) \arrow[d, "\approx"] \\ \pi_k^{st}(\Sigma^\infty X \wedge E) \arrow[r, "(f_*, g_*)"] & \pi_k^{st}(\Sigma^\infty Y \wedge E) \oplus \pi_k^{st}(\Sigma^\infty Z \wedge E) \arrow[r, "(h_*, -k_*)"] & \pi_k^{st}(\Sigma^\infty W \wedge E) \end{tikzcd}$$

Next we use the property that $\pi_k(\Omega^\infty A) = \pi_k^{st}(A)$, so that the bottom row of the diagram (11) is naturally isomorphic to

$$\pi_k(\Omega^\infty(\Sigma^\infty X \wedge E)) \xrightarrow{(f_*, g_*)} \pi_k(\Omega^\infty(\Sigma^\infty Y \wedge E)) \oplus \pi_k(\Omega^\infty(\Sigma^\infty Z \wedge E)) \xrightarrow{(h_*, -k_*)} \pi_k(\Omega^\infty(\Sigma^\infty W \wedge E)).$$

Of course this is just

$$\pi_k(F(X)) \xrightarrow{(f_*, g_*)} \pi_k F(Y) \oplus \pi_k F(Z) \xrightarrow{(h_*, -k_*)} \pi_k F(W).$$

To finish this proof we use the property of the homotopy pullback that $F(X)$ maps to the homotopy pullback P of

$$\begin{tikzcd} & F(Y) \arrow[d] \\ F(Z) \arrow[r] & F(W) \end{tikzcd}$$

together with Lemma 12 and the 5-lemma to see that $F(X)$ is weakly equivalent to P, and thus that the square

$$\begin{tikzcd} F(X) \arrow[r, "f"] \arrow[d, "g"] & F(Y) \arrow[d, "h"] \\ F(Z) \arrow[r, "k"] & F(W) \end{tikzcd}$$

is a homotopy pullback square. □

Lemma 12. *Let X be the homotopy pullback of*

$$\begin{tikzcd} & Y \arrow[d, "h"] \\ Z \arrow[r, "k"] & W \end{tikzcd}$$

with maps $f : X \to Y$ and $g : X \to Z$. Then there is a long exact sequence in homotopy

$$\cdots \to \pi_k(X) \xrightarrow{(f_*,g_*)} \pi_k(Y) \oplus \pi_k(Z) \xrightarrow{(h_*,-k_*)} \pi_k(W) \xrightarrow{\delta} \pi_{k-1}(X).$$

Proof. Recall from definition 9 that

$$X = \{(y, z, \omega) \in Y \times Z \times W^I : h(y) = \omega(0), k(z) = \omega(1)\}.$$

This means that

$$\begin{tikzcd} X \arrow[r, "\pi_3"] \arrow[d, "(\pi_1 \times \pi_2)"] & W^I \arrow[d, "(\omega(0),\omega(1))"] \\ Y \times Z \arrow[r, "(h,k)"] & W \times W \end{tikzcd}$$

is a regular pullback square. The right hand column is a fibration with fiber ΩW (look at the fiber over (w_0, w_0)) and the pullback of a fibration is a fibration with the same fiber, we get that the map $X \to Y \times Z$ is a fibration with fiber ΩW.

Note that π_1 is the f from the statement, and π_2 is the g from the statement. So we get the long exact sequence of the fibration

$$\pi_{k+1}(W) \to \pi_k(X) \xrightarrow{(f_*,g_*)} \pi_k(Y) \oplus \pi_k(Z) \xrightarrow{(h_*,-k_*)} \pi_k(W) \to \cdots$$

□

2.2.1. *The functor T_1 and the construction of P_1F.* Given a homotopy functor $F : \underline{\mathbf{Spaces}}_* \to \underline{\mathbf{Spaces}}_*$ we want to construct an excisive approximation to F: $P_1F : \underline{\mathbf{Spaces}}_* \to \underline{\mathbf{Spaces}}_*$, with a natural transformation $p_1F : F \to P_1F$. This needs to satisfy two properties.

1. P_1F needs to be excisive.
2. It needs to satisfy the following universal property: given a natural transformation $\eta : F \to G$ to a homotopy functor G, there is a unique natural transformation (up to weak homotopy equivalence) $\eta' : P_1F \to G$ such that $\eta = \eta' \circ p_1F$.

We first define an approximation to P_1F, which we'll call T_1F. We'll make use of the join functor:

$$X * U = (X \times U \times I)/\sim \text{ where } (x,u,0) \sim (x,u',0) \text{ and } (x,u,1) \sim (x',u,1),$$

primarily when U is a finite discrete set. When U is empty, $X * U = X$. When U is a single point, $X * U$ is the cone on X. When U is a pair of points, $X * U$ is the suspension of X. When U is k points, $X * U$ is k copies of the cone of X, identified at the open ends.

Definition 13. We define the functor T_1F and the natural transformation $t_1F : F \to T_1F$ by

$$F(X) = F(X * \emptyset) \xrightarrow{t_1F} (T_1F)(X) := \operatorname{holim} \left(\begin{tikzcd} & F(X * \{1\}) \arrow[d] \\ F(X * \{2\}) \arrow[r] & F(X * \{1,2\}) \end{tikzcd} \right)$$

Proposition 14. *If F is reduced (see Definition 7) then*

$$(T_1F)(X) \simeq_w \Omega F(\Sigma X).$$

Proof. Since $X * \{1\}$ and $X * \{2\}$ are contractible, the upper right and lower left of the partial square that we are taking the holim of are weakly equivalent to $F(*)$. Since F is reduced, then $F(*)$ is weakly contractible, so $(T_1F)(X)$ is weakly equivalent to the homotopy limit of

$$\begin{tikzcd} & \{*\} \arrow[d] \\ \{*\} \arrow[r] & F(\Sigma X). \end{tikzcd}$$

□

For the definition of P_1F, we need the homotopy colimit of a sequence of spaces.

Definition 15. Suppose we have a sequence of spaces

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots$$

We wish to define the homotopy colimit of this sequence of spaces. Recall that if $f : X \to Y$, $\mathrm{Cyl}(f) = [Y \vee (X \times I)]/\sim$ where $(x,1) \sim f(x)$ and $(x_0,t) \sim y_0$.

Y is a s.d.r. of $\mathrm{Cyl}(f)$, and if we include X into $\mathrm{Cyl}(f)$ by taking x to $(x, 0)$ then there is a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "="] \arrow[d] & \mathrm{Cyl}(f) \arrow[d, "r"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

where r is the strong deformation retract, and $X \to \mathrm{Cyl}(f)$ is a cofibration in the category of pointed spaces.

To make the homotopy colimit we replace X_2 with $X_2' = \mathrm{Cyl}(f_1)$, X_3 with $X_3' = \mathrm{Cyl}(f_2 \circ r)$, X_4 with $X_4' = \mathrm{Cyl}(f_3 \circ r)$ and so on.

Now the maps from $X_n' \to X_{n+1}'$ are all inclusions (in fact cofibrations). We define the hocolim to be the union of all the X_n'.

Exercise 2. Prove that if we have a sequence of maps $g_n : X_n \to Y$ so that $g_{n+1} \circ f_n \simeq g_n$ for each n then there is a map $g : \mathrm{hocolim}\{X_n\} \to Y$ so that $g|_{X_n} = g_n$.

Exercise 3. Prove that

$$\pi_k(\mathrm{hocolim}\{X_n\}) = \mathrm{colim}\, \pi_k(X_n).$$

Definition 16. We define P_1F by

$$(P_1F)(X) = \mathrm{hocolim}(F(X) \xrightarrow{t_1F} (T_1F)(X) \xrightarrow{t_1(T_1F)} (T_1^2F)(X) \to \cdots).$$

Note that this gives $p_1F : F \to (P_1F)$ from the map of $F(X)$ into the colimit.

Note also that T_1^2F is $T_1(T_1F)$, that is the construction T_1 applied to the functor T_1F.

Proposition 17. *If F is reduced,*

$$(P_1F)(X) = \mathrm{hocolim}(F(X) \to \Omega F(\Sigma X) \to \Omega^2 F(\Sigma^2 X) \to \cdots) = \Omega^\infty \underline{F}(X).$$

Here $\underline{F}(X)$ is the spectrum

$$\{F(X), F(\Sigma X), F(\Sigma^2 X), \ldots\}$$

with structure maps given by the adjoints of the maps $F(Y) \to (T_1F)(Y) \simeq \Omega F(\Sigma Y)$.

Proof. This is proved by combining the definition of (P_1F) with Proposition 14. □

Proposition 18. *If F is excisive, then $F \xrightarrow{t_1F} T_1F$ is a weak equivalence.*

Proof. Consider the diagram

$$\begin{tikzcd}
X * \emptyset \arrow[r] \arrow[d] & X * \{1\} \arrow[d] \\
X * \{2\} \arrow[r] & X * \{1,2\}
\end{tikzcd}$$

Since it is a homtopy pushout and F is excisive, the diagram

$$\begin{tikzcd}
F(X * \emptyset) \arrow[r] \arrow[d] & F(X * \{1\}) \arrow[d] \\
F(X * \{2\}) \arrow[r] & F(X * \{1,2\})
\end{tikzcd}$$

is a homotopy pullback. So the map $t_1 F : F(X) \to (T_1 F)(X)$ is a weak equivalence by the long exact sequence in homotopy of a homotopy pullback. $\square$

Exercise 4.

$$P_1(F \circ \Sigma) \cong (P_1 F) \circ \Sigma.$$

Hint: You need to check (and use) that $\Sigma(X * U) \cong (\Sigma X) * U$.

Remark. $(P_1 F)$ depends only on objects arbitrarily close to $\{*\}$. Suppose X is m-connected. Note that $X * U$ is $m+1$-connected (unless U is empty). So $(T_1 F)(X)$ is determined by the value of F on $m+1$-connected objects.

$(T_1^2(F)) = (T_1(T_1 F))(X)$ is determined by $(T_1 F)$ on $m+1$-connected objects $(X * U)$, which is in turn determined by F on $m+2$-connected objects.

In general, $T_1^i F = (T_1(T_1^{i-1} F))(X)$ is determined by $T_1^{i-1} F$ on $m+1$-connected objects, so by induction is determined by F on $m+i$-connected objects.

So $(P_1 F)(X)$ is determined by $m+i$-connected object for arbitrarily high i. If we define two spaces X, Y as being “close” if there is a highly connected map $X \to Y$, it follows that $(P_1 F)(X)$ is determined by the value of F on objects arbitrarily close to $\{*\}$!

Note that this is analogous to the linear approximation of a function $f(x)$ at $x = 0$. There is an analogy to the lineary approximation at $x = a$. It involves working with the domain category of spaces over a fixed space A.

2.3. Properties of $P_1 F$.

2.4. **General homotopy colimits and limits.** The standard reference for general homotopy limits (and colimits) is Bousfield and Kan's book *Homotopy limits, completions and localizations* [2]. They refer to homotopy limits as “homotopy inverse limits” [2, XI] and to homotopy colimits as “homotopy direct limits” [2, XII]. Bousfield and Kan treat the case where the category is simplicial sets instead of spaces, but this is not an important difference - their constructions work in a great deal of generality.

2.4.1. *Realization of a simplicial space.* We denote the standard n-simplex by Δ_n with vertices indicated by $\{0, 1, \ldots, n\}$. For each $i = 0, \ldots n+1$ there are face maps

$$d^i : \Delta_n \to \Delta_{n+1}$$

define by $d^i(j) = j$ for $j < i$ and $d^i = j + 1$ for $j \geq i$ and degeneracy maps

$$s_i : \Delta_n \to \Delta_{n-1}$$

for $0 \leq i \leq n-1$ by $s_i(j) = j$ for $j \leq i$ and $s_i(j) = j - 1$ for $j > i$.

Definition 19. Let $\{X_n\}$ be a simplicial space. The simplicial space is a sequence of space X_n that come with face maps for $0 \leq i \leq n$

$$d_i : X_n \to X_{n-1}$$

and degeneracy maps for $0 \leq i \leq n-1$

$$s^i : X_{n-1} \to X_n$$

The realization functor is

$$|X_\bullet| = [\coprod_{i=0}^{\infty} (X_i \times \Delta_i)] / \sim$$

where $(x, d^i(t)) \sim (d_i(x), t)$ and $(x, s_i(t)) = (s^i(x), t)$.

There is a standard construction associated to any small category I that gives us a simplicial set (this is a simplicial space but with each X_n simply a discrete space).

Definition 20. The *nerve* of a small category I is the simplicial set where the 0-simplices are the objects, the 1-simplices are given by all morphisms, the 2-simplices are given by all pairs of composable morphisms and the n-simplices are given by all n-tuples of composable morphisms.

$$N(I)_n = \{a_0 \xleftarrow{f_1} a_1 \xleftarrow{f_2} \cdots \xleftarrow{f_n} a_n : a_i \in \mathrm{ob}(I), f_i \in \mathrm{Hom}_I(a_i, a_{i-1})\}$$

The boundary maps

$$d_i : N(I)_n \to N(I)_{n-1}$$

is defined by

$$\begin{aligned} d_0(f_1, \ldots, f_n) &= (f_2, \ldots, f_n) \\ d_i(f_1, \ldots, f_n) &= (f_1, \ldots, f_{i-1}, f_i \circ f_{i+1}, f_{i+2}, \ldots, f_n) \text{ and} \\ d_n(f_1, \ldots, f_n) &= (f_1, \ldots, f_{n-1}). \end{aligned}$$

The degeneracy maps

$$s^i : N(I)_{n-1} \to N(I)_n$$

are given by

$$s^i(f_1, \ldots, f_{n-1}) = (f_1, \ldots, f_{i-1}, 1_{a_{i-1}}, f_i, \ldots, f_{n-1}).$$

So the degeneracy maps insert an identity into the string of composable maps, the face maps either compose two maps, or they drop a map off of the end. Commonly one thinks of face maps as omitting an object. This makes them analogous to face maps on ordinary simplicial complexes where the ith face is the face opposite to the ith vertex.

Remark. In this case one has a slightly simpler description of the realization.

$$|N(I)_\bullet| = \left(\coprod_{i=0}^{\infty} \overline{N}(I)_i \times \Delta_i \right) / \sim$$

where $\overline{N}(I)_i$ consists of the composable strings of morphisms of length i in which no identity maps occur (this is the complement of the image of the degeneracy maps) and then the relation $\sim$ can be given by

$$(x, d_i(t)) = (d^i(x), t).$$

If we try to explicitly understand then how $|N(I)_\bullet|$ is built, we start with vertices for each object of I. Then for each non-identity morphism of I we get a 1-simplex from the target to the source (a loop if the target and source happen to coincide).

Then a 2-simplex is attached for each composable pair of non-identity morphisms. One edge is attached to each of the pair of morphisms, and the third edge is attached to the composite. This provides a homotopy from the 1-simplex corresponding to the composition to the pair of edges corresponding to the pair of maps. One can think of this as parametrizing the ways to think of a composition.

Given a triple of composable morphisms (f_1, f_2, f_3) we get two simplices for the pairs $(f_1 \circ f_2, f_3), (f_1, f_2 \circ f_3), (f_1, f_2)$ and (f_2, f_3). The 3-simplex corresponding to (f_1, f_2, f_3) gets attached to these 4 faces. One can think of this as paremtrizing the ways to think of a composition of 3 maps. The face (f_2, f_3) (which corresponds to $f_2 \circ f_3$) is the face opposite to the vertex corresponding to the domain of f_1 and the face (f_1, f_2) (corresponding to $f_1 \circ f_2$) is the one opposite to the vertex corresponding to the range of f_3.

2.4.2. *Definition of homotopy colimit.* We suppose that I is a small category and $G : I \to \underline{\mathbf{Spaces}}_*$ is a functor. We define a simplicial space $G_\bullet$ by

$$G_n = \bigvee_{(f_1,\ldots,f_n)\in N(I)_n} \operatorname{dom}(f_n)$$

$$\begin{aligned} d_0 x_{(f_1,\ldots,f_n)} &= x_{(f_2,\ldots,f_n)} \\ d_i x_{(f_1,\ldots,f_n)} &= x_{(f_1,\ldots,f_{i-1},f_i\circ f_{i+1},f_{i+2},\ldots,f_n)} \text{ and} \\ d_n x_{(f_1,\ldots,f_n)} &= f_n(x)_{(f_1,\ldots,f_{n-1})}. \end{aligned}$$

Here I'm using a subscript to indicate which summand in the one point union the point belongs to. So the d_i all simply change the summand except for d_n.

The s are defined similarly - they just change the summand, and none of them change the point in the summand.

Definition 21. We define

$$\operatorname{hocolim} G = \operatorname*{hocolim}_I G = |G_\bullet|$$

At this point there is so much abstraction raveled up in this definition that I'm afraid it seems very obscure. To make it less obscure one should think about examples where the category I is very small.

Let's take the example where $I = P'(\underline{2})$. Recall this has objects $\{\emptyset, \{1\}, \{2\}\}$ with morphisms give by inclusion. In this case, $N(I)$ has very few non-degenerate simplices. There are three 0-simplices (one for each object) and two non-degenerate 1-simplices for each non-trivial inclusion. All other simplices contain at least one incidence of an identity map, so that points in them get identified with points in lower dimensional simplices. The identifications in $|N(I)_\bullet|$ give

$$(\emptyset, \Delta_0) = (d_0(\emptyset \to \{1\}), \Delta_0) \sim (\emptyset \to \{1\}, d^0(\Delta_0)) = (\emptyset \to \{1\}, 1) \in (\emptyset \to \{1\}, [0,1])$$

(here we deliberately confuse Δ_0 and the point it contains, and we consider Δ_1 to be the interval $[0,1]$). Similarly,

$$(\emptyset, \Delta_0) = (d_0(\emptyset \to \{2\}), \Delta_0) \sim (\emptyset \to \{2\}, d^0(\Delta_0)) = (\emptyset \to \{2\}, 1) \in (\emptyset \to \{2\}, [0,1]).$$

We similarly get

$$(\{1\}, \Delta_0) = (d_1(\emptyset \to \{1\}), \Delta_0) \sim (\emptyset \to \{1\}, d^1(\Delta_0)) = (\emptyset \to \{1\}, 0) \in (\emptyset \to \{1\}, [0,1])$$

and

$$(\{2\}, \Delta_0) = (d_1(\emptyset \to \{2\}), \Delta_0) \sim (\emptyset \to \{2\}, d^1(\Delta_0)) = (\emptyset \to \{2\}, 0) \in (\emptyset \to \{2\}, [0,1]).$$

This exhausts the non-degenerate simplicies in $N(I)$, so we see that $|N(I)_\bullet|$ is two intervals attached at one end.

Now let $G : P'(\underline{2}) \to \underline{\mathbf{Spaces}}_*$ be a functor. We want to work out $\operatorname{hocolim}_{P'(\underline{2})} G$. We'll write $X = G(\emptyset)$, $Y = G(\{1\})$, $Z = G(\{2\})$, f for the map $X \to Y$ and g for the map $X \to Z$. To construct the homotopy colimit, it will again be sufficient to look at non-degenerate simplices. For the non-degenerate 0 simplices $N(I)$ we get

$$X, Y \text{ and } Z \text{ indexed respectively by } \emptyset, \{1\} \text{ and } \{2\}. \tag{22}$$

For the non-degenerate 1-simplices we get

$$X \text{ and } X \text{ indexed by } \{1\} \xleftarrow{\alpha_1} \emptyset \text{ and } \{2\} \xleftarrow{\alpha_2} \emptyset.$$

In realizing G we need to cross with simplicies of the appropriate dimensions and do the appropriate identifications. So

$$|G_\bullet| = [(X \vee Y \vee Z) \sqcup (X \times I \vee X \times I)] \,/ \sim$$

where Δ_0 is omitted completely from the notation since it is the one point space, and I is used for Δ_1. Remember that the two copies of $X \times I$ are indexed by the two different non-identity morphisms, α_i. To do the identifications we let $x \in X$, and I'll use a subscript on each element of X, Y or Z to remind us what simplex from $N(I)$ it is associated to.

$$x_\emptyset = d_0(x_{\{2\}\leftarrow\emptyset}) \sim (x_{\{2\}\leftarrow\emptyset}, d^0(\Delta_0)) = (x_{\{2\}\leftarrow\emptyset}, 1).$$

$$x_\emptyset = d_0(x_{\{1\}\leftarrow\emptyset}) \sim (x_{\{1\}\leftarrow\emptyset}, d^0(\Delta_0)) = (x_{\{1\}\leftarrow\emptyset}, 1).$$

$$f(x)_{\{1\}} = d_1(x_{\{1\}\leftarrow\emptyset}) \sim (x_{\{1\}\leftarrow\emptyset}, d^1(\Delta_0)) = (x_{\{1\}\leftarrow\emptyset}, 0)$$

$$g(x)_{\{2\}} = d_1(x_{\{2\}\leftarrow\emptyset}) \sim (x_{\{2\}\leftarrow\emptyset}, d^1(\Delta_0)) = (x_{\{2\}\leftarrow\emptyset}, 0).$$

So the two copies of $X \times I$ are both glued to X at the 0 ends (making a cylinder on X with length 2) and the respective 1-ends are glued to either Y by f or to Z by g. This is obviously homeomorphic to the homotopy pushout described in Definition4.

It is instructive to do a similar pair of analyses with the category $P'(\underline{3})$. We begin with the nerve of this category. The objects of the category are

$$\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}$$

with morphisms given by inclusions. So there are 7 0-simplices in $N(P'(\underline{3}))$, 12 non-degenerate 1-simplices, and 6 non-degenerate 2-simplices. Each 2-simplex corresponds to a sequence of morphisms

$$\emptyset \to \{i\} \to i, j$$

for i in $\{1,2,3\}$ and $j \neq i$ in $\{1,2,3\}$ The realization is a hexagon with the vertex $\emptyset$ in the center, and the other vertices around the edges to that each $\{i,j\}$ vertex is between vertices $\{i\}$ and $\{j\}$. One can think about this hexagon as what one would get if one took the surface of the cube, removed the 3 faces that contain the vertex $\{1,2,3\}$ since $\{1,2,3\} \notin P(\underline{3})$, subdivided the remaining 3 faces into triangles by joining the vertex $\emptyset$ with the opposite vertex, and then spreading out the surface onto the plane.

The homotopy colimit of G is probably best pictured by imagining a space over each point of the hexagon. Here we'll label spaces by $X_U = G(U)$ as U runs through $P'(\underline{3})$, and we'll try to avoid naming the maps. For p in the interior of a simplex, the space over p should be the domain associated to that simplex as in the definition of the simplicial space $G_\bullet$. Except around the outer edges of the hexagon then, the relevant space is $X_\emptyset$. Over each vertex, the space is $X_{\{i\}}$ or $X_{\{i,j\}}$, matching the vertex. Over each interior point of the outer edge, the space is $X\{i\}$ corresponding to the vertex $\{i\}$ on that edge.

The space that results (the homotopy colimit) contains

$$X_{\{1,2\}} \sqcup X_{\{1,3\}} \sqcup X_{\{2.3\}}$$

as a subspace (the subspace over those three outer vertices). The subspace $X_i \times I$ (over the two half-open outer edges around the vertex $\{i\}$) is attached at the ends to $X_{\{i,j\}}$ and $X_{\{i,k\}}$ for the other two integers j and k by the maps $X_i \to X_{\{i,j\}}$ and $X_i \to X_{\{i,k\}}$ respectively. So the subspace over the two outer edges attached to $\{1\}$ for example is the homotopy pushout of

$$\begin{tikzcd}
X_{\{1\}} \arrow[r] \arrow[d] & X_{\{1,2\}} \\
X_{\{1,3\}}. &
\end{tikzcd}$$

The subspace over the entire outer ring is the union of the three obvious homotopy pushouts where the two occurences of $X_{\{1,2\}}$ are identified (and the same for $X_{\{1,3\}}$ and $X_{\{2,3\}}$).

To understand the entire homotopy colimit, we take the union of the subspace over the outer edge with the product

$$X_{\emptyset} \times H$$

where H is the hexagon. At the edge of the hexagon, we identify $X_{\emptyset}$ with its image in the appropriate space.

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