The Tate spectrum of v_n-periodic complex oriented theories

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Abstract

Let G be a finite group. We prove that the Tate spectrum with respect to G of the n^{th} Morava K-theory spectrum $K(n)$ is equivariantly contractible. As a corollary we can deduce results about the Tate spectrum with respect to G of various v_n-periodic homology theories. For example, if K_n is the integral Morava K-theory spectrum, then $t_G(K_n)$ is rational. This generalizes the result for $n = 1$ that appears as [11, Theorem 16.1].

More succinctly but less precisely, if E is complex oriented and v_n-periodic then $t_G(E)$ is complex oriented and v_{n-1}-periodic.

Our results are related to and partially motivated by certain conjectures about the relationship between Mahowald's root invariant and v_n-periodic homotopy.

1 Introduction

In this paper we work primarily in the equivariant stable category of [16], but we try to provide numerous reference points and nonequivariant versions of results for readers unfamiliar with that world.

We discuss the *Tate spectrum* $t_G(X)$. X is a G-spectrum and $t_G(X)$ is a new G-spectrum, covariantly functorial in X. Throughout this paper we

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assume that G is a finite group (though the functor $t_G(X)$ is defined whenever G is compact Lie).

Our introduction is in three parts; first we review a bit of equivariant stable homotopy theory from [16], then we briefly discuss the Tate spectrum of [11], and then we state our results. The rest of the paper is organized as follows. In Section 2 we prove some lemmas required for our Theorem 1.1 (and the theorem itself when G is cyclic), and in Section 3 we prove our main theorem, Theorem 1.1 for arbitrary finite G. Section 4 applies Theorem 1.1 to prove the remainder of our results about the Tate theory of complex oriented v_n-periodic spectra.

1.1 G-spectra

Recall that a G-spectrum X is a set of spaces indexed over G-invariant finite dimensional subspaces of some complete G-universe U. (U is an infinite dimensional G invariant inner product space containing a countably infinite direct sum of regular representations of G as a subspace.) The usual structure maps

$$X(V) \xrightarrow{\cong} \Omega^W X(V \oplus W)$$

are required to be G-maps. Here V and W are disjoint finite dimensional G-subspaces of U.

There are two functors we will use to move between the equivariant and nonequivariant worlds. First we have

$$i_* : \{\text{spectra}\} \to \{G\text{-spectra}\},$$

which comes from the inclusion

$$i : U^G \hookrightarrow U$$

of a (nonequivariant) universe into U. This functor is defined by taking i_*X to be the spectrum derived by turning the prespectrum

$$V \longmapsto S^{W^\perp} \wedge X(W)$$

into a spectrum. Here $W = V \cap U^G$ and $W^\perp$ is the orthogonal complement of W in V.

On the space level, we may take any space X and regard it as a $G-$space with the trivial G-action. The stable analogue is the functor that takes an ordinary spectrum and gives the G-spectrum $i_*(X)$. (Of course G doesn't really act trivially on i_*X since it acts non-trivially on the space associated to V in the prespectrum defined above whenever V is a non-trivial representation of G.) The same construction gives a functor from the category of naive G-spectra (G-spectra indexed over a universe on which G acts trivially) to the category of G-spectra. One can think of i_* as the composition of the inclusion functor from non-equivariant spectra to naive G-spectra with the construction described above which constructs a G-spectrum from a naive G spectrum. The functor i_* thought of as starting in naive G-spectra is left adjoint to the appropriate forgetful functor, and the inclusion of non-equivariant spectra into the category of naive G-spectra is left adjoint to taking G-fixed points.

The other functor

$$j^* : \{G\text{-spectra}\} \to \{\text{spectra}\}$$

comes from forgetting the G-structure. We look at U as an $\{e\}$-universe via the inclusion $j : \{e\} \to G$, and look at each G-space in a G-spectrum as an $\{e\}$-space.

One would like to have the following equality

$$j^* i_* X = X.$$

Unfortunately, this doesn't actually make any sense since the two spectra are indexed over different universes. The usual way to get around this is to let f denote the linear isometric embedding

$$U^G \to j^* U$$

(here j^*U is U with the G-action forgotten) and then it follows from chasing the definitions that

$$j^* i_* X = f_* X.$$

Now from [16, II.1.5] it follows that f_* gives an isomorphism of the homotopy categories of spectra over the universes U^G and j^*U.

All the ideas presented above about G-spectra are taken from [16], where of course much more information and detail are present.

1.2 The Tate spectrum

The Tate spectrum is the main subject of [11]. We recall some important relevant properties here. Let X be a G-spectrum. Then we have another G-spectrum $F(EG_+, X)$, the spectrum of "maps" from EG_+ to X. G acts by conjugation with the maps. Recall the cofiber sequence of G-spaces (G acts trivially on S^0) from [6]

$$EG_+ \to S^0 \to \tilde{E}G.$$

The first map sends the base point to the base point and everything else to the other point; $\tilde{E}G$ is defined to be the cofiber. We are now ready to define

$$t_G(X) \overset{\text{def}}{=} F(EG_+, X) \wedge \tilde{E}G. \tag{1}$$

This spectrum comes with a map from X:

$$X \to F(EG_+, X) \to F(EG_+, X) \wedge \tilde{E}G$$

which is a map of ring spectra when X is a ring spectrum [11, Proposition 3.5]. Note that if F, F' are finite CW-spectra, and $f : F \to F'$ then

$$F(Y, X \wedge F) \simeq F(Y, X) \wedge F$$

and

$$F(Y, 1_X \wedge f) \simeq F(Y, 1_X) \wedge f.$$

It follows then from (1) that

$$t_G(X \wedge F) \simeq t_G(X) \wedge F \text{ and } t_G(1_X \wedge f) \simeq t_G(1_X) \wedge f. \tag{2}$$

We will also use the spectrum

$$f_G(X) \simeq X \wedge EG_+. \tag{3}$$

We will make frequent use of [11, Proposition 2.6], which states that

$$t_G(X) \simeq F(\tilde{E}G, \Sigma f_G(X)). \tag{4}$$

Corollary 2.4 below [11, 16.1] gives a description of the nonequivariant spectrum

$$t_G(i_* X)^G$$

when G is a finite cyclic group. This description connects the Tate spectrum with Mahowald's inverse system

$$\cdots \to (B\mathbf{Z}/(p))_{-r} \to (B\mathbf{Z}/(p))_{-r+1} \to \cdots$$

in the case $G = \mathbf{Z}/(p)$; in that case, Corollary 2.4 can be rephrased to say

$$t_{\mathbf{Z}/(p)}(i_*X)^{\mathbf{Z}/(p)} = \varprojlim_r (\Sigma X \wedge (B\mathbf{Z}/(p))_{-r}). \tag{5}$$

We use $\varprojlim$ to denote the homotopy inverse limit. (For a definition of $(B\mathbf{Z}/(p))_{-r}$ see [24].)

1.3 Results

In this paper we are concerned with $t_G(E)$ when E is a periodic homology theory and G is finite. Essentially, we only know how to deal with $E = i_*K$ where K is a nonequivariant periodic homology theory, but [11, Corollary 1.5] enables us to extend our results. We may say that E is an equivariant form of K if E is a split G-spectrum with underlying spectrum $j^*E = K$; by [11, Corollary 1.5] our results apply equally well to any equivariant form of K, and from the equivariant point of view, i_*K is often a rather artificial equivariant form of K. For example, if K is ordinary K-theory, the best equivariant form is the classical equivariant K-theory formed from equivariant bundles, and this is quite different from i_*K. Nevertheless, for clarity we work with the particular equivariant form i_*K for the remainder of the paper.

Our main theorem is the following.

Theorem 1.1 *Suppose $k = i_*K(n)$ where $K(n)$ is the n^{th} Morava K-theory spectrum. Then*

$$t_G(k) \simeq *.$$

Jack Morava points out the following consequence about the transfer. Recall from [11, §5] that $t_G(k)$ is the cofiber of the obvious composite

$$k \wedge EG_+ \to k \to F(EG_+, k).$$

Assume $k = i_*K$ with K a naive G-spectrum on which G acts trivially. Then taking fixed points, the cofiber of the composite

$$K \wedge_G EG_+ \xrightarrow{\tilde{\tau}} (i_*K \wedge EG_+)^G \to i_*K^G \to F(EG_+, i_*K)^G \simeq F(BG_+, K)$$

is $t_G(i_*K)^G$, where $\tilde{\tau}$ (which is a homotopy equivalence) is the adjoint of the equivariant transfer $i_*K \wedge_G EG_+ \to i_*K \wedge EG_+$ (see also [1]). If $K = K(n)$, Theorem 1.1 gives the following.

Corollary 1.2 *The composite of the adjoint of the transfer with the inclusion into the homotopy fixed points gives an equivalence*

$$K(n) \wedge BG_+ \to F(BG_+, K(n)).$$

In other words, the transfer gives a duality map making BG_+ self-dual with respect to $K(n)$.

The next results describe what effect Tate homology has on certain types of periodic complex oriented spectra.

Let

$$M = M(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$$

be a finite spectrum with

$$BP_*(M) = BP_*/(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}}).$$

Such spectra are guaranteed to exist for sufficiently large multi-indices $I = (i_0, \ldots, i_{n-1})$ by the periodicity theorem of [13].

Definition 1.3 *We call E v_n-*periodic *if E is complex oriented and v_n is a unit on $E \wedge M(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}})$.*

One such example is the Johnson-Wilson theory $E(n)$. Recall that $E(n)$ is a complex oriented theory with

$$E(n)_* = \mathbf{Z}_{(p)}[v_1, \ldots, v_{n-1}, v_n, v_n^{-1}].$$

The orientation factors through BP, and each v_i is the image of $v_i \in BP_*$. This example is v_n-periodic since v_n is actually a unit on the spectrum $E(n)$, but it is clear that the condition in Definition 1.3 is less restrictive.

Definition 1.3 is independent of the choice of multi-index I and of the spectrum $M(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}})$.

By integral Morava K-theory, we mean a spectrum K such that K is complex oriented, has no torsion in its homotopy groups, and reduces to $K(n)$ modulo p. The examples we have in mind are $E(n)/(v_1, \ldots, v_{n-1})$ with coefficients $\mathbf{Z}_{(p)}[v_n, v_n^{-1}]$ or the p-completion of that spectrum (with coefficients $\mathbf{Z}_p[v_n, v_n^{-1}]$).

Corollary 1.4 *If $E = i_*K$, where K is a p-local integral Morava K-theory, then $t_G(E)$ is rational.*

We use the notation $X^\wedge_{I_k}$ for the completion of X with respect to the ideal $(p, \ldots, v_{k-1}) = I_k \subset BP_*$. More precisely, the construction is

$$X^\wedge_{I_k} = \lim_{\overleftarrow{I}} (X \wedge M(p^{i_0}, \ldots, v_{k-1}^{i_{k-1}})),$$

where the inverse limit is taken over maps

$$M(p^{j_0}, \ldots, v_{n-1}^{j_{n-1}}) \to M(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}})$$

commuting with inclusion of the bottom cell. Such maps are easily constructed courtesy of the nilpotence theorem of [9] (see for example [19, Proposition 3.7] for existence of these maps and some uniqueness properties).

In the case $X = E(n)$ this completion agrees with the Artinian completion of [3], and for various MU-modules this completion can be constructed using Baas-Sullivan manifolds with singularities [2] or using the recent work of Elmendorf, Kriz, and May on E_∞ modules. It is a straightforward corollary of the nilpotence theorem that the construction given here coincides with Bousfield localization with respect to any of the $M(p^{i_0}, \ldots, v_{k-1}^{i_{k-1}})$ (a proof of this appears in [14]), hence the construction is well defined regardless of any choices.

We note the following generalization of Corollary 1.4.

Proposition 1.5 *Let $E = i_*K$ where K is complex oriented and v_n acts as a unit on K and some power of the ideal*

$$(v_{i+1}, \ldots, v_{n-1})$$

acts trivially. Then v_i acts as a unit on $t_G(E)^\wedge_{I_i}$.

If some power of

$$(v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{n-1})$$

acts trivially, then v_i is a unit on $t_G(E)$.

We compare Bousfield classes (see [5] for notation). The hypotheses of Proposition 1.5 imply that $\langle j^*E \rangle \leq \langle E(n) \rangle$, and the conclusion implies $\langle t_G(E)^H \rangle \leq \langle E(n-1) \rangle$ for all $H \leq G$.

We note the following corollaries of Proposition 1.5.

Corollary 1.6 *If $E = i_*K$, where K is complex oriented and such that v_n acts as a unit on K, then v_{n-1} acts as a unit on $(t_G(E))^{\wedge}_{I_{n-1}}$.*

In particular Corollary 1.6 holds when $E = i_*E(n)$.

Corollary 1.7 *If $E = i_*K$, where K is a v_n-periodic spectrum, then $t_G(E)$ is v_{n-1}-periodic.*

It is worth explicitly noting that the preceding results are all false for $G = S^1$. For E complex oriented, Theorem 16.1 of [11] implies

$$t_{S^1}(E)^{S^1} \simeq \underleftarrow{\lim}_r \bigvee_{k \geq -r} \Sigma^{2k} E.$$

By, for example, perusal of the Atiyah-Hirzebruch spectral sequence for $E_*(\mathbf{C}P^\infty)$ and related spectra, one can check that the maps

$$\bigvee_{k \geq -r} \Sigma^{2k} E \to \bigvee_{k \geq -r+1} \Sigma^{2k} E$$

in the inverse system to compute $t_{S^1}(E)^{S^1}$ can be taken to be the projection maps. It follows that for any integer a

$$\begin{aligned} t_{S^1}(E)^{S^1} \simeq \underleftarrow{\lim}_r \bigvee_{k \geq -r} \Sigma^{2k} E &\simeq (\bigvee_{k \geq a} \Sigma^{2k} E) \vee (\underleftarrow{\lim}_r \bigvee_{k=r}^{k=a-1} \Sigma^{2k} E) \\ &\simeq \bigvee_{k \geq a} \Sigma^{2k} E \vee \prod_{k<a} \Sigma^{2k} E. \end{aligned}$$

Our work can be encapsulated by the slogan "Tate homology reduces chromatic periodicity." It was motivated by the observation of [11] that $t_G(KU_G)$ is rational when G is finite, and by work centered on the Mahowald invariant (also called the root invariant, see [8, 7, 21, 20, 24]) where the corresponding slogan, a conjecture due to Mahowald and Ravenel [18], is "the Mahowald invariant converts v_{n-1}-periodic homotopy to v_n-periodic homotopy."

The Mahowald invariant associates a coset in $\pi_*(X)$ to each element of

$$\pi_*(\underleftarrow{\lim}_r (\Sigma X \wedge (B\mathbf{Z}/(p))_{-r})) = \pi_*(t_{\mathbf{Z}/(p)}(i_*X)^{\mathbf{Z}/(p)}) \tag{6}$$

(see equation (5)). When X is finite, the spectrum in equation (6) is just the p-completion of X, by Lin's theorem [17, 12].

Although Mahowald and Ravenel's conjecture was only meant to apply to finite spectra, when X is a v_n periodic spectrum (in the sense of Definition 1.3) $t_{\mathbf{Z}/(p)}(i_* X)^{\mathbf{Z}/(p)}$ is v_{n-1}-periodic by Corollary 1.7, so this gives examples consistent with Mahowald and Ravenel's conjecture.

It is beyond the scope of this paper, but work of Hopkins and Ravenel, together with Theorem 1.1, can be used to show that if X is $E(n)$ local, then $t_G(i_* X)^G$ is $E(n-1)$ local. When $G = \mathbf{Z}/(p)$ (in which case Theorem 1.1 was already known), this can be thought of as supplying more examples consistent with Mahowald and Ravenel's conjecture.

We thank Jack Morava for initiating these questions, and for introducing the second author to Tate spectra. The first author gratefully acknowledges the support of the Nuffield Foundation, and both authors thank the *Mathematisches Forschungsinstitut Oberwolfach* where this work began.

2 Inverse limits of Thom spectra

Suppose ξ is a vector bundle over X. From the embedding of vector bundles

$$\begin{array}{ccc} E & \longrightarrow & E' \\ {\scriptstyle s\xi}\downarrow & & \downarrow{\scriptstyle (s+1)\xi} \\ X & \xrightarrow{=} & X \end{array}$$

we get a map of Thom spectra $X^{(s\xi)} \to X^{(s\xi+\xi)}$.

We remind the reader that the Thom spectrum $X^{(s\xi)}$ can be defined (as a spectrum, not a space) even when s is negative. The construction is due to Boardman [4] and is also discussed in [16, IX] and [25]. Briefly, if X is compact, find an n large enough so that $s\xi \oplus [n]$ is a vector bundle ($[n]$ is the n-dimensional trivial bundle). Then

$$X^{(s\xi)} = \Sigma^{-n} X^{(s\xi \oplus [n])}.$$

For more general spaces X, we can take the direct limit of the Thom spectra of the compact subcomplexes.

It is clear from the above discussion that the map

$$X^{(s\xi)} \to X^{(s\xi+\xi)}$$

is defined when s is negative as well as positive, so we get an inverse system

$$\cdots \to X^{(s\xi)} \to X^{(s\xi+\xi)} \to \cdots$$

for $s \in \mathbf{Z}$. We need the following nonequivariant result ($K = K(n)$ is the motivating example in Lemmas 2.1 and 2.2).

Lemma 2.1 *Let ξ be a positive dimensional vector bundle over BG, K a complex oriented theory with $K_*(BG)$ finitely generated over K_*. Then*

$$\varprojlim_s (K \wedge BG^{(-s\xi)}) = *.$$

Proof: The finite generation implies that all the classes in $K_*(BG)$ are supported on some finite skeleton, say $BG^{\langle r\rangle}$.

It is then clear that the map $BG^{\langle r\rangle} \to BG$ is surjective on K_*, and hence by the Thom isomorphism the map

$$K \wedge (BG^{\langle r\rangle})^{(-s\xi)} \xrightarrow{i} K \wedge BG^{(-s\xi)} \tag{7}$$

is surjective on homotopy for all s.

Let λ be the dimension of ξ. If we choose j so that $r - (s+j)\lambda < -s\lambda$ (i.e. $r < j\lambda$) then the composite

$$K \wedge (BG^{\langle r\rangle})^{(-(s+j)\xi)} \xrightarrow{i} K \wedge BG^{(-(s+j)\xi)} \xrightarrow{e} K \wedge BG^{(-s\xi)}$$

is null since the top cell of the $(BG^{\langle r\rangle})^{(-(s+j)\xi)}$ is in a lower dimension than the bottom cell of $BG^{(-s\xi)}$.

But the map i is onto in homotopy by (7), so the map e is 0 on homotopy groups. Since e is just a composite of maps in the inverse system under investigation, it follows that the homotopy of the inverse limit is 0. □

We use this to deduce the following result in the category of G-spectra.

Lemma 2.2 *Let V be some finite, positive dimensional representation of G. Let K be a complex oriented theory such that $K_*(BH)$ is finitely generated over K_* for all $H \leq G$. Then the spectrum*

$$F(S^{\infty V}, i_*K \wedge EG_+)$$

is equivariantly contractible.

Proof: For any spectrum K,

$$\begin{aligned} F(S^{\infty V}, i_* K \wedge EG_+) &\simeq \lim_{\overleftarrow{r}} F(S^{rV}, i_* K \wedge EG_+) \\ &\simeq \lim_{\overleftarrow{r}} i_* K \wedge EG_+ \wedge S^{-rV}. \end{aligned}$$

If $H \leq G$ then V is of course also a representation of H, and EG is an EH. So

$$(i_* K \wedge EG_+ \wedge S^{-rV})^H \simeq K \wedge BH^{(-r\xi)}. \tag{8}$$

Here ξ is the bundle over BH induced by the H-representation V, and $BH^{(-r\xi)}$ is the obvious Thom spectrum. The equivalence (8) is by [16, Corollary II.7.2], combined with the identification

$$(EG_+ \wedge S^{-rV})/H = BH^{(-r\xi)}$$

from [22]. Equation (8) is also discussed in [22]; in Theorem B the square

$$\begin{array}{ccc} (i_* K \wedge EG_+ \wedge S^{-rV})^H & \xrightarrow[\simeq]{} & K \wedge BH^{(-r\xi)} \\ \downarrow & & \downarrow \\ (i_* K \wedge EG_+ \wedge S^{(-r+1)V})^H & \xrightarrow[\simeq]{} & K \wedge BH^{([-r+1]\xi)} \end{array}$$

is shown to commute (the left hand vertical map is induced by the map $S^0 \to S^V$, and the right hand map is induced by the evident map of Thom spectra).

From this we see that

$$F(S^{\infty V}, i_* K \wedge EG_+)^H \simeq \lim_{\overleftarrow{r}} K \wedge BH^{(-r\xi)},$$

and the right hand side is contractible by Lemma 2.1. Since this is true for all $H \leq G$, we can apply the G-Whitehead theorem to $F(S^{\infty V}, i_* K \wedge EG_+)$. □

Ravenel [23] proves that $K(n)^*(BG)$ is finitely generated over the graded field $K(n)^*$. By duality it follows that $K(n)_*(BG)$ is finitely generated over $K(n)_*$. Hence $K = K(n)$ satisfies the hypotheses of Lemmas 2.1 and 2.2 for all finite groups G. We do not know any significantly different examples.

Corollary 2.3 $t_G(i_*K(n)) \simeq *$ *when* G *is a cyclic group.*

Proof: Take V to be the cyclic representation of G on $\mathbf{C}$, ξ to be the corresponding complex line bundle over BG.

Then EG_+ is $S(\infty V)_+$, and $\tilde{E}G = S^{\infty V}$. So by (3) and (4):

$$t_G(i_*K(n)) \simeq F(S^{\infty V}, i_*K(n) \wedge \Sigma EG_+)$$

and the right hand side is equivariantly contractible by Lemma 2.2. □

We note that the proof of Lemma 2.2 is the same as that used in [11] to arrive at the following result.

Corollary 2.4 ([11, Theorem 16.1]) *If* $H < G$ *is cyclic and* K *is a non-equivariant spectrum then*

$$t_G(i_*K)^H \simeq \lim_{\substack{\longleftarrow \\ r}} (\Sigma K \wedge BH^{(-r\xi)}).$$

(Here ξ *is the complex line bundle over* BH *induced by a faithful action of* H *on* $\mathbf{C}$*.)*

Proof: Let V a faithful representation of H on $\mathbf{C}$. Then

$$\begin{aligned} t_G(i_*K)^H &\simeq t_H(i_*K)^H \\ &\simeq F(S^{\infty V}, (i_*\Sigma K) \wedge EH_+)^H \\ &\simeq \lim_{\substack{\longleftarrow \\ r}} [(i_*\Sigma K) \wedge EH_+ \wedge S^{-rV}]^H. \end{aligned}$$

The second equivalence is from equation (4). □

3 Proof of Theorem 1.1

Proposition 3.1 *If* K *is a complex oriented theory with* $K_*(BG_+)$ *finitely generated over* K_* *for all* G*, then* $t_G(i_*K) \simeq *$ *for all* G*.*

We delay the proof until after the statement of Proposition 3.2.

Proposition 3.2 *If W is a non-zero, finite dimensional G-representation with $W^G = 0$, and k is a G-spectrum such that for every $H \lneq G$, $t_H(k) \simeq *$, then*
$$F(S^{\infty W}, \Sigma f_G(k)) \simeq t_G(k).$$

Proof of Proposition 3.1: Now by Lemma 2.2 if we take a non-zero finite dimensional G-representation W, such that $W^G = \{0\}$ (for example the reduced regular representation) then

$$F(S^{\infty W}, f_G(i_* K)) \simeq F(S^{\infty W}, i_* K \wedge EG_+) \simeq *.$$

So Proposition 3.1 now follows from Proposition 3.2. □

Proof of Proposition 3.2: By our hypotheses, when $H \lneq G$

$$t_H(k)^H \simeq F(\tilde{E}G, \Sigma f_G(k))^H \simeq *$$

(the first equality is by [11, Proposition 2.6], i.e. equation (4)). Also, $F(S^{\infty W}, f_G(k))^H \simeq *$ because if $W^H = 0$ we can apply Proposition 3.2 inductively, and if $W^H \neq 0$ then $S^{\infty W}$ is contractible.

So it suffices by the G-Whitehead theorem [16, Theorem I.5.10] to produce a map
$$F(S^{\infty W}, f_G(k)) \to F(\tilde{E}G, f_G(k))$$
that is an isomorphism on G-fixed points. Our proof is a variation of the proof of part (b) of [6, Theorem A].

We smash $S^{\infty W}$ with the cofibration

$$EG_+ \to S^0 \to \tilde{E}G. \tag{9}$$

The G-spectrum $S^{\infty W} \wedge EG_+$ is equivariantly contractible by the G-Whitehead theorem. This is because if
$$\{e\} \lneq H \leq G$$
then $(EG_+)^H = *$, hence

$$(S^{\infty W} \wedge EG_+)^H = *.$$

On the other hand, $S^{\infty W}$ is nonequivariantly contractible, so $(S^{\infty W})^{\{e\}} \simeq *$, therefore
$$(S^{\infty W} \wedge EG_+)^{\{e\}} \simeq *.$$

It follows then from smashing with (9) that

$$S^{\infty W} \to S^{\infty W} \wedge \tilde{E}G \tag{10}$$

is a G-equivalence.

As spaces, $(S^{\infty W})^G = S^0$, so

$$(S^{\infty W}/S^0)^G \simeq *,$$

therefore $(S^{\infty W}/S^0)$ can be built from G-cells of the form $(G/H)_+ \wedge E^n$ where $H \underset{\neq}{<} G$. Now we have the following isomorphisms. The first is by [16, Proposition II.4.3], and the last is the inductive hypothesis.

$$\begin{aligned} F(G/H_+ \wedge S^n \wedge \tilde{E}G, f_G(k))^G &\cong F(S^n \wedge \tilde{E}G, f_G(k))^H \\ &\simeq \Sigma^{-n-1} t_H(k)^H \simeq *. \end{aligned}$$

It follows by taking the limit over skeleta of $S^{\infty W}/S^0$ that

$$F((S^{\infty W}/S^0) \wedge \tilde{E}G, f_G(k))^G \simeq *. \tag{11}$$

Therefore we get a map

$$\begin{aligned} F(S^{\infty W}, f_G(k)) &\simeq F(S^{\infty W} \wedge \tilde{E}G, f_G(k)) \\ &\to F(S^0 \wedge \tilde{E}G, f_G(k)) \\ &\simeq F(\tilde{E}G, f_G(k)) \end{aligned}$$

the first equivalence following from (10).

If we apply G-fixed points to this composite, we get an equivalence by (11). □

Theorem 1.1 follows from Proposition 3.1.

4 Tate spectra of v_n-periodic spectra

Proof of Corollary 1.4. Let $E = i_* K$ be a spectrum as in the hypothesis.

We have a cofibration sequence

$$K \xrightarrow{p} K \to K(n).$$

Since t_G preserves cofibrations as is obvious from the definition, we get (see equation (2))

$$t_G(i_*K) \xrightarrow{p} t_G(i_*K) \to *,$$

so the $\cdot p$ map on $t_G(i_*K)$ is an equivalence. Now by [11, Proposition 1.1] it follows that the $\cdot p$ map on $t_G(E)$ is also an equivalence. Hence $t_G(E)$ is rational. □

Note that this proof clearly works for the appropriate integral theories as well as the p-local ones, for example with complex K-theory, KU, in place of the Adams summand of the p-local theory.

The proof also applies to KO. We use the notation $M(2)$ for the $\mathbf{Z}/(2)$ Moore spectrum with bottom cell in dimension 0 and $C(\eta)$ for the mapping cone on $\eta : S^1 \to S^0$. Recall that

$$KU = KO \wedge C(\eta)$$

so

$$KU \wedge M(2) = KO \wedge C(\eta) \wedge M(2).$$

So we have the cofibration

$$\Sigma KO \wedge M(2) \xrightarrow{\eta} KO \wedge M(2) \to KU \wedge M(2).$$

Now $t_G(i_*[KU \wedge M(2)]) \simeq *$ since $KU \wedge M(2) = K(1)$, so

$$t_G(i_*[\Sigma KO \wedge M(2)]) \xrightarrow{\eta} t_G(i_*[KO \wedge M(2)])$$

is an equivalence. But η is nilpotent, so $t_G(i_*[KO \wedge M(2)]) \simeq *$. Then the proof of Corollary 1.4 shows $t_G(i_*KO)$ is rational.

Now we wish to prove Proposition 1.5. We first require the following.

Proposition 4.1 *Suppose k is a split G-spectrum whose underlying spectrum is a p-local v_n-periodic spectrum with v_n a unit and v_i nilpotent for $0 \le i \le n-1$ ($v_0 = p$). Then $t_G(k) \simeq *$.*

Proof: As usual there is no loss of generality in assuming our spectrum is i_*K with underlying spectrum K. We have already proven this proposition for $K = K(n)$. Let $B(n)$ be the spectrum derived from BP that satisfies

$$B(n)_* = \mathbf{Z}/(p)[v_n, v_n^{-1}, v_{n+1}, v_{n+2}, \ldots,]$$

Remark 6.19 of [26] implies that $B(n)_*(BG)$ is free over $B(n)_*$ on the same number of generators as is required to generate $K(n)_*(BG)$ over $K(n)_*$. By [23] this is a finite number whenever G is finite. So by Proposition 3.1, the proposition follows for $K = B(n)$.

Now let K be any spectrum satisfying our hypotheses. For each $j < n$, there is some i_j so that $v_j^{i_j}$ is null on K. For sufficiently large i_j, the spectra

$$M(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}})$$

discussed in the introduction (as well as the spectra $M(p^{i_0}, \ldots, v_j^{i_j})$ for $j < n-1$) can be taken to be ring spectra by [10].

By the assumption that i_j is large enough so that $v_j^{i_j}$ is null on K, $K \wedge M$ is finite wedge of copies of K. So if $t_G(i_*K \wedge M) \simeq *$, then $t_G(i_*K) \simeq *$.

$K \wedge M$ is also a module spectrum over $BP \wedge M$, and hence over $v_n^{-1}BP \wedge M$. So $t_G(i_*K \wedge M)$ is a module spectrum over $t_G(i_*v_n^{-1}BP \wedge M)$.

We need to know a little about $v_n^{-1}BP \wedge M$. We write $BP/(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}})$ for the iterated cofiber by v_i-maps induced by the BP-module structure. This spectrum is usually constructed from BP (or directly from MU) by appealing to Baas-Sullivan theory [2]. Such spectra are discussed in [2], and also in papers such as [15]. (The recent work of Elmendorf, Kriz and May on E_∞ module spectra also gives a construction of the $BP/(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}})$.)

Now with M as above,

$$BP \wedge M = BP/(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}}).$$

This can easily be checked by induction. Both spectra are iterated cofibers. Assuming

$$BP \wedge M(p^{i_0}, \ldots, v_j^{i_j}) = BP/(p^{i_0}, \ldots, v_j^{i_j})$$

the maps whose cofibers form the next stage are both induced by the same homotopy class in the ring spectrum $BP \wedge M(p^{i_0}, \ldots, v_j^{i_j})$.

Now if $i_j > 1$ there is a cofibration of BP-module spectra

$$\begin{aligned}\Sigma^{2p^j-2}BP/(p^{i_0}, \ldots, v_j^{i_j-1}, \ldots, v_{n-1}^{i_{n-1}}) \xrightarrow{v_j} & BP/(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}}) \\ & \to BP/(p^{i_0}, \ldots, v_j, \ldots, v_{n-1}^{i_{n-1}}).\end{aligned}$$

So by induction on $i_0 + \cdots + i_{n-1}$, $BP \wedge M$ has a finite filtration (of length $i_0 + \cdots + i_{n-1}$) such that the cofibers are suspensions of $BP/(p, \ldots, v_{n-1})$.

It follows that $v_n^{-1}BP \wedge M$ has a finite filtration where the cofibers are copies of $B(n)$, so $t_G(i_* v_n^{-1}BP \wedge M) \simeq *$. Hence $t_G(i_* K \wedge M) \simeq *$. □

Remark: It seems quite likely that under the hypotheses of Proposition 4.1 $K_*(BG)$ is always finitely generated over K_* when G is finite, but the proof in [23] does not generalize immediately to this situation.

Proof of Proposition 1.5: We again use spectra

$$M(p^{j_0}, v_1^{j_1}, \ldots, v_{i-1}^{j_{i-1}}, v_i^{j_i})$$

as in the proof of Proposition 4.1, with BP-homology isomorphic to

$$BP_*/(p^{j_0}, v_1^{j_1}, \ldots, v_{i-1}^{j_{i-1}}, v_i^{j_i}).$$

As before, we use the fact that such spectra exist for a cofinal indexing set and can be taken to be ring spectra by [10].

Then

$$E \wedge M(p^{j_0}, \ldots, v_i^{j_i})$$

is a spectrum satisfying the hypotheses of Proposition 4.1, so

$$t_G(E \wedge M(p^{j_0}, \ldots, v_i^{j_i})) = t_G(E) \wedge M(p^{j_0}, \ldots, v_i^{j_i}) \simeq *.$$

It follows that v_i is an equivalence on

$$t_G(E \wedge M(p^{j_0}, \ldots, v_{i-1}^{j_{i-1}})) = t_G(E) \wedge M(p^{j_0}, \ldots, v_{i-1}^{j_{i-1}}).$$

But

$$t_G(E)^{\wedge}_{I_i} = \underleftarrow{\lim}\,(t_G(E) \wedge M(p^{j_0}, \ldots, v_{i-1}^{j_{i-1}}))$$

so v_i is an equivalence on $t_G(E)^{\wedge}_{I_i}$. □

Proof of Corollary 1.7: By Proposition 1.5 v_{n-1} acts as a unit on

$$t_G(E \wedge M(p^{i_0}, \ldots, v_{n-2}^{i_{n-2}})),$$

which by equation (2) is equivalent to $t_G(E) \wedge M(p^{i_0}, \ldots, v_{n-2}^{i_{n-2}})$. □

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