TATE COHOMOLOGY LOWERS CHROMATIC BOUSFIELD CLASSES

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ABSTRACT. Let G be a finite group. We use the results of [5] to show that the Tate homology of $E(n)$ local spectra with respect to G produces $E(n-1)$ local spectra. We also show that the Bousfield class of the Tate homology of $L_n X$ (for X finite) is the same as that of $L_{n-1}X$.

To be precise, recall that Tate homology is a functor from G-spectra to G-spectra. To produce a functor P_G from spectra to spectra, we look at a spectrum as a naive G-spectrum on which G acts trivially, apply Tate homology, and take G-fixed points. This composite is the functor we shall actually study, and we'll prove that $\langle P_G(L_n X)\rangle = \langle L_{n-1}X\rangle$ when X is finite.

When $G = \Sigma_p$, the symmetric group on p letters, this is related to a conjecture of Hopkins and Mahowald (usually framed in terms of Mahowald's functor $\mathbf{R}P_{-\infty}(-)$).

1. INTRODUCTION

We briefly recall the spectra that occur in Lin's proof of the Segal conjecture for the group $\mathbf{Z}/(2)$. Embed $\mathbf{Z}/(p)$ into S^1 and look at the pullback of the tautological (complex) line bundle over BS^1 as a bundle over $B\mathbf{Z}/(p)$. Call this bundle ξ.

We denote by P_{-2k} the spectrum given by the Thom spectrum $(B\mathbf{Z}/(p))^{-k\xi}$ when $p = 2$, or when p is odd the summand of that spectrum corresponding to $B\Sigma_p$. (We refer the reader to [20] for the definition of a Thom spectrum associated to a virtual bundle). P_{-2k} has a cell in every dimension $\geq -2k$ and is the spectrum freqeuently called $\mathbf{R}P^{\infty}_{-2k}$ when $p = 2$. When p is odd, P_{-2k} has a cell in every dimension congruent to 0 or -1 modulo $q = 2p-2$ and $\geq -2k$. P_{-2k} is the same as the spectrum denoted P^{∞}_{-2k} in [19], and constructed there by James periodicity rather than via Thom spectra.

Lin's theorem [10] (Gunarwardena's theorem when $p > 2$ [1]) states that

$$\varprojlim_k (P_{-2k} \wedge X) = \Sigma^{-1} X^{\wedge}_p \tag{1}$$

when X is a finite spectrum. This inverse system of spectra (with some minor alterations) is also what is used to define the root invariant (see [11] for $p = 2$, or more generally, [13, 19]). As shorthand, we write $P_{-\infty}(X)$ for $\varprojlim_k (P_{-2k} \wedge X)$.

Mahowald and Ravenel [12] have conjectured a relationship between chromatic periodicity and Mahowald's root invariant. There is a related conjecture by Hopkins and Mahowald that is more closely related to our concerns in this paper. Denote

Date: December 7, 1994.
1991 *Mathematics Subject Classification.* 55P60, 55P42; Secondary 55N91.
The authors were partially supported by the NSF..

Bousfield localization with respect to $E(n)$ by L_n. They conjecture that

$$P_{-\infty}(L_n X) = \Sigma^{-2} L_{n-1} X\hat{_p} \vee \Sigma^{-1} L_{n-1} X\hat{_p} \tag{2}$$

for X finite. (A word to the experts: this conjecture is connected with Hopkins's chromatic splitting conjecture (see [7]) and at the time (2) was conjectured the chromatic splitting conjecture was in too simple a form. In light of its current corrected form as in [7], (2) is probably also too optimistic, though we expect it is true as stated when X has type $n-1$.)

Greenlees and May put the $P_{-\infty}$ construction in a more general context in [4]. There they define the Tate G spectrum associated to a G spectrum X, $t_G(X)$. We are not concerned with equivariant spectra here, but we use t_G to construct a functor from (ordinary) spectra to spectra. We abuse notation and write i_* for the functor which is the composite of the inclusion of ordinary spectra into the category of naive G-spectra (as the objects on which G acts trivially) with the left adjoint of the forgetful functor from G-spectra to naive G-spectra. So i_* is a functor from ordinary spectra to G-spectra. Our other functor is the G-fixed point functor, $(-)^G$ which goes from G-spectra to spectra. We refer the reader to [9] for details. We define

$$P_G(X) = t_G(i_* X)^G.$$

Then [4, 16.1] shows that

$$P_{\Sigma_p}(X) = P_{-\infty}(\Sigma X).$$

(the left hand side needs to be localized at p when p is odd).

Henceforth we will only be concerned with the case where G is a finite group. We now have a family of functors, one for each finite group G, and we assume (to guarantee non-triviality of our functors) that p divides the order of G. Our main theorem is the following.

Theorem 1.1. *Let X be a finite spectrum. Then $\langle P_G(L_n X)\rangle = \langle L_{n-1} X\rangle$*

Here $\langle X\rangle$ means the Bousfield class of the spectrum X as given in [2].

We also give a result about complex oriented v_n-periodic spectra.

Theorem 1.2. *If E is Landweber exact and v_n-periodic then $P_{\mathbf{Z}/(p)}E$ is Landweber exact and v_{n-1}-periodic. It follows generally that $\langle P_G(E_{(p)})\rangle = \langle E(n-1)\rangle$.*

Our proofs rely on [5, Theorem 1.1] which implies that $P_G(K(n)) \simeq *$. We also use two other results that are relatively well known. We use Ravenel's Proposition 1.34 from [16]:

$$\langle X\rangle = \langle C(f)\rangle \vee \langle Tel(f)\rangle \tag{3}$$

where f is a self map of X, $C(f)$ is the cofiber and $Tel(f)$ is the mapping telescope. Finally, we use a theorem of Hopkins and Ravenel from [18] to show that $P_G(K(n)) \simeq *$ implies $P_G(L_n X) \simeq *$ when X is finite type n.

Using the interesting results of Mahowald and Shick in [15] one can show that $P_{\mathbf{Z}/(2)}(Tel(X)) \simeq *$ where X is finite type n and $Tel(X)$ is the mapping telescope of X under a v_n map. One can use this to deduce our theorems in the special case $G = \mathbf{Z}/(2)$. Chun-Nip Lee has done this independently [8].

The authors would like to thank Neil Strickland for pointing out various places where we failed to say what we meant or mean what we said in a previous draft

2. There is a finite type n spectrum F with L_nF $K(n)$-nilpotent

Recall that X is said to be E-prenilpotent if L_EX is E-nilpotent; that is if L_EX can be built up in finitely many stages from spectra of the form $E \wedge Z$ by taking cofibrations and retracts. Note that N is E-nilpotent implies $N \wedge M$ is $E \wedge M$-nilpotent. By [18, 8.3] there is a finite type 0 spectrum Y that is L_nBP-prenilpotent.

By [18, Lemma 8.1.4]

$$\langle L_nBP \rangle = \langle v_n^{-1}BP \rangle$$

which is in turn equal to $\langle E(n) \rangle$ by [16]. So L_nY is L_nBP-nilpotent. Now let M be a finite type n spectrum with

$$BP_*M = BP_*/(p^{i_0}, \dots , v_{n-1}^{i_{n-1}})$$

and such that M is a ring spectrum. (See [3] for the existence of such ring spectra.) Then $BP \wedge M = BP/(p^{i_0}, \dots , v_{n-1}^{i_{n-1}})$, so by [17, Theorem 1],

$$L_nBP \wedge M = v_n^{-1}BP \wedge M.$$

But $v_n^{-1}BP \wedge M$ is made out of finitely many cofibrations with cofiber $v_n^{-1}BP/I_n = B(n)$. Now by [21, Remark 6.19], $B(n) = K(n) \wedge B$ for some B, so it follows that $L_nBP \wedge M$ is $K(n)$-nilpotent.

Since $L_nY \wedge M$ is $L_nBP \wedge M$-nilpotent it follows that $L_nY \wedge M$ is $K(n)$-nilpotent.

Lemma 2.1. *There is a finite type n spectrum F with L_nF $K(n)$-nilpotent.*

Proof. Take $F = Y \wedge M$. □

3. $P_G(L_nX \wedge F) \simeq *$ if F is type n.

We recall from [5] that $t_G(i_*K(n)) \simeq *$ as a G-spectrum. It follows that $P_G(K(n)) \simeq *$. We record the following lemma.

Lemma 3.1. *If X is $K(n)$-nilpotent then $P_G(X) \simeq *$.*

Proof. First note that since P_G takes cofibrations to cofibrations, it suffices to prove that $P_G(K(n) \wedge Z) \simeq *$ for any Z. But $P_G(R)$ is a ring spectrum when R is a ring spectrum, and $P_G(N)$ is a module spectrum over $P_G(R)$ if N is a module spectrum over R [4, Proposition 3.5]. It follows that $P_G(K(n) \wedge Z) \simeq *$. □

Remark: The same proof shows that $t_G(X) \simeq *$ equivariantly if X is $i_*K(n)$-nilpotent in the category of G-spectra.

Corollary 3.2. *If F is finite type n, then $P_G(L_nX \wedge F) \simeq *$ for any spectrum X.*

Proof. Let $\mathcal{C}$ be the category of finite spectra F such that $P_G(L_nX \wedge F) \simeq *$ for all spectra X. $\mathcal{C}$ is a thick subcategory in the sense of [6]. It follows that if $\mathcal{C} \cap \mathcal{C}_n \neq \emptyset$ then $\mathcal{C}_n \subseteq \mathcal{C}$. We recall that if a spectrum Y is $K(n)$-nilpotent, so is $X \wedge Y$ for any spectrum X. Since $L_nX \wedge F = X \wedge L_nF$ (L_n is smashing) by Lemma 3.1 and Lemma 2.1, $\mathcal{C} \cap \mathcal{C}_n \neq \emptyset$. □

Remark: Since $L_nF = L_{K(n)}F$ when F is finite type n, one might ask when $P_G(L_{K(n)}X) \simeq *$. While we don't know the most general answer, this does not hold in general for X finite. Using the methods of this paper, one can easily check that if X is finite, $\langle P_G(L_{K(n)}X) \rangle = \langle L_{n-1}X \rangle$.

4. $P_G(L_nX)$ is $E(n-1)$-local.

We use equation (3) inductively. We get

$$\begin{aligned}\langle P_G L_n X\rangle &= \langle p^{-1}P_G L_n X\rangle \vee \langle P_G L_n X \wedge M(p^{i_0})\rangle \\ &= \langle p^{-1}P_G L_n X\rangle \vee \langle v_1^{-1}P_G L_n X \wedge M(p^{i_0})\rangle \vee \cdots \\ &\quad \vee\langle v_{n-1}^{-1}P_G L_n \wedge M(p^{i_0},\ldots, v_{n-2}^{i_{n-2}})\rangle \vee \langle P_G L_n X \wedge M(p^{i_0},\ldots, v_{n-1}^{i_{n-1}})\rangle.\end{aligned}$$

Now since the $P_G L_n X \wedge M(p^{i_0},\ldots, v_{n-1}^{i_{n-1}}) \simeq *$ by Corollary 3.2, we get

$$\begin{aligned}\langle P_G L_n X\rangle &= \langle p^{-1}P_G L_n X\rangle \vee \langle v_1^{-1}P_G L_n X \wedge M(p^{i_0})\rangle \vee \cdots \\ &\quad \vee\langle v_{n-1}^{-1}P_G L_n \wedge M(p^{i_0},\ldots, v_{n-2}^{i_{n-2}})\rangle.\end{aligned}$$

Since $P_G L_n X$ is an $L_n S^0$ module, it follows that

$$\begin{aligned}v_j^{-1}P_G L_n X \wedge M(p^{i_0},\ldots, v_{j-1}^{i_{j-1}}) &= P_G L_n X \wedge v_j^{-1} L_n M(p^{i_0},\ldots, v_{j-1}^{i_{j-1}}) \\ &= (P_G L_n X) \wedge L_j M(p^{i_0},\ldots, v_{j-1}^{i_{j-1}}).\end{aligned}$$

(The last equality follows from [14, Proposition 6.1].) Since $\langle L_j M(p^{i_0},\ldots, v_{j-1}^{i_{j-1}})\rangle = \langle K(j)\rangle$, we see that

$$\langle P_G L_n X\rangle \leq \langle K(0)\rangle \vee \cdots \vee \langle K(n-1)\rangle = \langle E(n-1)\rangle. \tag{4}$$

Now since $L_n X$ is a $L_n S^0$-module, $P_G L_n X$ is a $P_G L_n S^0$ module [4, Proposition 3.5]. But $P_G L_n S^0$ is self local since it is a ring spectrum [16], so by equation (4) $P_G L_n S^0$ is $E(n-1)$-local, hence so is $P_G L_n X$.

To finish the proof of Theorem 1.1 it remains to show the inequality in equation (4) is actually an equality when $X = S^0$. In section 6 we use Theorem 1.2 to do this.

5. P_G of Landweber exact v_n-periodic theories.

In this section we prove Theorem 1.2. We will suppose that E is a complex oriented homology theory. We also assume that E is p-local ($P_G(E) = P_G(E_{(p)}$ if G happens to be a p-group). Then we can assume E is oriented by a map from BP, so that we can consider v_i as an element of E_*. We remind the reader that $I_j = (p, v_1, \ldots, v_{j-1})$, and that for BP (and hence for any spectrum oriented from BP),

$$[p](x) = px +_F v_1 x^p +_F v_2 x^{p^2} +_F \cdots +_F v_i x^{p^i} +_F \cdots$$

where $+_F$ is the sum in the formal group law on E_*.

We begin by remarking that for complex oriented E in which the leading coefficient of $[p](x)$ is not a 0-divisor,

$$\pi_* P_{\mathbf{Z}/(p)} E = E_*((x))/([p](x))$$

where $|x| = -2$, $E_*((x))$ denotes the ring of Laurent series over E_* which have only finitely many terms involving negative powers of x, and $[p](x)$ is the p-series. It follows that when $[p](x)$ is not a zero divisor, we have a short exact sequence

$$E_*((x)) \xrightarrow{\cdot [p](x)} E_*((x)) \to \pi_* P_{\mathbf{Z}/(p)} E.$$

We now assume that E is v_n-periodic Landweber exact. We define v_n-periodic almost as in [5, Definition 1.3]; E is v_n-periodic if v_n is a unit on E_*/I_n and in addition $E_*/I_n \neq 0$.

For each $j \leq n$, we know that $E_*/I_j \to v_j^{-1}E_*/I_j$ is injective by the hypothesis of Landweber exactness. It follows that $E_*((x))/I_j = E_*/I_j((x)) \to (v_j^{-1}E_*/I_j)((x))$ is injective also. Now $[p](x)$ is a unit in $(v_j^{-1}E_*/I_j)((x))$ since it is a power series with leading term $v_j x^{p^j}$, which is a unit. It follows that

$$E_*((x))/I_j \to (v_j^{-1}E_*/I_j)((x)) \xrightarrow{\cdot [p](x)} (v_j^{-1}E_*/I_j)((x))$$

is injective, hence

$$E_*((x))/I_j \xrightarrow{\cdot [p](x)} E_*((x))/I_j$$

is also.

We examine the diagram of short exact sequences below, in which the bottom row is the cokernel of the map between the top two rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E_*((x))/I_j & \xrightarrow{\cdot [p](x)} & E_*((x))/I_j & \longrightarrow & (\pi_* P_{\mathbf{Z}/(p)}E)/I_j & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \cdot v_j} & & \downarrow{\scriptstyle \cdot v_j} & & \downarrow{\scriptstyle \cdot v_j} & & \\
0 & \longrightarrow & E_*((x))/I_j & \xrightarrow{\cdot [p](x)} & E_*((x))/I_j & \longrightarrow & (\pi_* P_{\mathbf{Z}/(p)}E)/I_j & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E_*((x))/I_{j+1} & \xrightarrow{\cdot [p](x)} & E_*((x))/I_{j+1} & \longrightarrow & (\pi_* P_{\mathbf{Z}/(p)}E)/I_{j+1} & \longrightarrow & 0
\end{array}$$

By the snake lemma applied to the first two rows (together with the observation that the first two vertical maps are injective) we see that v_j is injective on $(\pi_* P_{\mathbf{Z}/(p)}E)/I_j$.

We can also see that $[p](x)$ is not a unit in $E_*((x))/I_j$ unless v_j is a unit in E_*/I_j, therefore $(\pi_* P_{\mathbf{Z}/(p)}E)/I_j \neq 0$ unless $j = n$, and this last observation tells us that v_{n-1} is a unit on $(\pi_* P_{\mathbf{Z}/(p)}E)/I_{n-1}$.

We conclude that $P_{\mathbf{Z}/(p)}E$ is Landweber exact, and that $\pi_*(P_{\mathbf{Z}/(p)}E)/I_{n-1} \neq 0$ while $\pi_*(P_{\mathbf{Z}/(p)}E)/I_n = 0$. It follows by using (3) as before (see [7, Corollary 1.12], that

$$\langle P_{\mathbf{Z}/(p)}E \rangle = \langle E(n-1) \rangle.$$

By using the maps of complex oriented ring spectra

$$E \to P_G E \to P_{\mathbf{Z}/(p)}E$$

(when $\mathbf{Z}/(p) \subseteq G$) we also deduce that

$$\langle P_G E \rangle = \langle E(n-1) \rangle.$$

6. Proof of Theorem 1.1.

We recall from section 4 that $v_j^{-1}P_G L_n S^0 \wedge M(p^{i_0}, \ldots, v_{j-1}^{i_{j-1}})$ has the Bousfield class of either a point or of $K(j)$. So to show it has the class of $K(j)$, we need only show that it is not contractible.

We pick $i_0, \ldots, i_{j-1}$ so that $M(p^{i_0}, \ldots, v_{j-1}^{i_{j-1}})$ is a ring spectrum. Then we observe that the map

$$S^0 \to v_j^{-1}P_G L_n S^0 \wedge M(p^{i_0}, \ldots, v_{j-1}^{i_{j-1}}) \to v_j^{-1}P_{\mathbf{Z}/(p)} L_n S^0 \wedge M(p^{i_0}, \ldots, v_{j-1}^{i_{j-1}}) \to$$

$$v_j^{-1}P_{\mathbf{Z}/(p)}E(n) \wedge M(p^{i_0}, \ldots, v_{j-1}^{i_{j-1}}) \xrightarrow{=} v_j^{-1}P_{\mathbf{Z}/(p)}E(n)/(p^{i_0}, \ldots, v_{j-1}^{i_{j-1}}) \to$$

$$v_j^{-1} P_{\mathbf{Z}/(p)} E(n)/(p, \dots, v_{j-1})$$

is the unit of the ring spectrum $v_j^{-1} P_{\mathbf{Z}/(p)} E(n)/(p, \dots, v_{j-1})$. This is non-zero if $j < n$ by Theorem 1.2. So none of the intervening spectra are contractible either.

For arbitrary finite X (instead of S^0) just smash with X. Note that $\langle - \rangle$, $P_G(-)$ and localization commute with smashing with a finite spectrum.

Remark: The same proof can be iterated to draw the obvious conclusions about $P_G^k(L_n S^0)$.

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