Math 342 **Solutions**
Midterm Exam

1. **True/False** If the statement is true, give a brief explanation; if it is false, provide a counterexample.

T **F** $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.

F $\mathbb{R}^2$ isn't even a sub*set* of $\mathbb{R}^3$.

T **F** Let A be an $m \times n$ matrix. If the equation $A\vec{x} = \vec{b}$ is consistent, then $\text{Col}(A) = \mathbb{R}^m$.

F Even without the typo, this is false. For example, if $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $A\vec{x} = \vec{b}$ is consistent, but $\text{Col}(A) \neq \mathbb{R}^2$.

T **F** If a finite set of nonzero vectors (call it S) spans the vector space V, then some subset of S is a basis for V.

T This is part of what Theorem 5 tells us.

T **F** The row space of A is the same as the column space of A^T.

T We think of $\text{Row}(A)$ as column vectors by 'standing up' the rows. That is, by identifying them with the columns of A^T.

T **F** A linearly dependent set of vectors in an n-dimensional vector space contains more than n vectors.

F The set $\{\vec{0}\}$ is dependent in any vector space of any dimension.

Problems

2. If the null space of a 4×7 matrix A is 3-dimensional, can you be sure that the transformation $\vec{x} \mapsto A\vec{x}$ is onto? Explain.

Solution: Yes. From the Rank Theorem we know that $\text{rk}(A) + \dim \text{Nul}(A) =$ number of columns of A. So $\text{rk}(A) + 3 = 7 \implies \text{rk}(A) = \dim \text{Col}(A) = 4$. Recall that $\text{Col}(A)$ is the range of $\vec{x} \mapsto A\vec{x}$. We see that the range of the transformation is a 4-dimensional subspace of $\mathbb{R}^4$. That means the range is all of $\mathbb{R}^4$, so the transformation is onto.

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3. If A is a 17×23 matrix, what is the smallest dimension the null space of A can have? Explain.

Solution: Well, $\text{rk}(A) + \dim \text{Nul}(A) =$ number of columns of A, so $\text{rk}(A) + \dim \text{Nul}(A) = 23$. Now $\text{rk}(A) =$ number of pivots of A, which can be no larger than the number of rows of A, since each pivot has a row to itself. The biggest $\text{rk}(A)$ can be is 17, so the smallest $\dim \text{Nul}(A)$ can be is $23 - 17 = 6$.

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4. Let $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : y = 2x - 1 \right\}$. Find a set of vectors that spans H or give a specific example to show that H is not a subspace of $\mathbb{R}^2$.

Solution: This is **not** a subspace of $\mathbb{R}^2$ since it does not contain the zero vector—$x = 0$ and $y = 0$ does not satisfy $y = 2x - 1$.

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5. Let $F = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Define a linear transformation $T : M_{2\times 2} \to M_{2\times 2}$ by $T(A) = FA - AF$. Describe the kernel and range of T. [Hint: Write A as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.]

Solution: Let's compute

$$T(A) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix} - \begin{bmatrix} b & a \\ d & c \end{bmatrix} = \begin{bmatrix} c-b & d-a \\ a-d & b-c \end{bmatrix}$$

This is the zero matrix (the zero vector of $M_{2\times 2}$) when $b = c$ and $a = d$. So

$$\ker(T) = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \ : \ a, b \in \mathbb{R} \right\}$$

Notice that $T(A) = \begin{bmatrix} c-b & d-a \\ a-d & b-c \end{bmatrix} = \begin{bmatrix} c-b & d-a \\ -(d-a) & -(c-b) \end{bmatrix}$

We see that

$$\text{range}(T) = \left\{ \begin{bmatrix} r & s \\ -s & -r \end{bmatrix} \ : \ r, s \in \mathbb{R} \right\}$$

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6. Let Q be a fixed invertible $n \times n$ matrix and define $T : M_{n\times n} \to M_{n\times n}$ be $T(A) = QAQ^{-1}$. Show that T is a linear transformation.

Solution: We have to check that T distributes over addition and scalar multiplication.

a) Let $A, B \in M_{n\times n}$. Then $T(A+B) = Q(A+B)Q^{-1} = (QA+QB)Q^{-1} = QAQ^{-1} + QBQ^{-1} = T(A) + T(B)$.

b) Let $A \in M_{n\times n}$ and let $r \in \mathbb{R}$. Then $T(rA) = Q(rA)Q^{-1} = rQAQ^{-1} = rT(A)$.

Hence, T is a linear transformation.

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7. Find bases for Nul$A$, Col$A$, and Row$A$ if

$$A = \begin{bmatrix} 1 & -2 & 2 & 3 & 1 \\ 2 & -4 & 5 & 9 & 1 \\ 3 & -6 & 4 & 3 & 5 \end{bmatrix}$$

Solution: Of course we row reduce.

$$\begin{bmatrix} 1 & -2 & 2 & 3 & 1 \\ 2 & -4 & 5 & 9 & 1 \\ 3 & -6 & 4 & 3 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -3 & 3 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot columns of A form a basis for Col(A). The pivot rows of an echelon form of A form a basis for Row(A). Our method for finding the solution space of $A\vec{x} = \vec{0}$ produces a basis for Nul(A).

A basis for Col(A) is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix} \right\}$

A basis for Row(A) is $(1, -2, 0, -3, 3), (0, 0, 1, 3, -1)$

A basis for Nul(A) is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

8. Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ and $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ be two bases for the vector space V, and suppose $\vec{b}_1 = 3\vec{c}_1 + 2\vec{c}_2$ and $\vec{b}_2 = 4\vec{c}_1 + 3\vec{c}_2$.

a) Find $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$

Solution: $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \left[\, [\vec{b}_1]_\mathcal{C} \quad [\vec{b}_2]_\mathcal{C} \,\right] = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$

b) If $\vec{x} = -\vec{b}_1 + 3\vec{b}_2$, find $[\vec{x}]_\mathcal{B}$.

Solution: $[\vec{x}]_\mathcal{B} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

c) If $\vec{x} = -\vec{b}_1 + 3\vec{b}_2$, find $[\vec{x}]_\mathcal{C}$.

Solution: $[\vec{x}]_\mathcal{C} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} [\vec{x}]_\mathcal{B} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix}$

d) If $[\vec{y}]_\mathcal{C} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, find $[\vec{y}]_\mathcal{B}$.

Solution: Since $[\vec{y}]_\mathcal{C} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} [\vec{y}]_\mathcal{B}$, we have

$$[\vec{y}]_\mathcal{B} = \left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \right)^{-1} [\vec{y}]_\mathcal{C} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 10 \\ -7 \end{bmatrix}$$