Math 342 — Solutions Quiz 2

1. The set $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$ is **not** a basis for $\mathbb{R}^3$. Find a subset of $\mathcal{S}$ that **is** a basis for $\mathbb{R}^3$.

Solution: Well, Span $\mathcal{S} = \mathrm{Col}(A)$, where $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & -1 & 1 & 1 \\ 3 & 1 & 4 & -1 \end{bmatrix}$, and we know how to find a basis for a column space.

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & -1 & 1 & 1 \\ 3 & 1 & 4 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The first, second and fourth columns are pivot columns, so the first second and fourth columns of A form a basis for $\mathrm{Col}(A)$. That is, a basis for $\mathbb{R}^3$ is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

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2. The set $\mathcal{B} = \{1 + t + t^2, 1 - t, 1 + 2t + 2t^2\}$ is a basis for $\mathbb{P}_2$.

a) Suppose $[p(t)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$. What is $p(t)$?

Solution: The coordinate vector tells us how to build p from the basis. That is,
$p(t) = 2(1 + t + t^2) + 3(1 - t) + (-1)(1 + 2t + 2t^2) = 4 - 3t.$

b) Let $q(t) = 3 - t + 2t^2$. Find $[q(t)]_{\mathcal{B}}$.

Solution: Here we have to find c_1, c_2, c_3 such that $c_1(1+t+t^2)+c_2(1-t)+c_3(1+2t+2t^2) = 3 - t + 2t^2$. This is $(c_1 + c_2 + c_3) + (c_1 - c_2 + 2c_3)t + (c_1 + 2c_3)t^2 = 3 - t + 2t^2$. We need

$$\begin{cases} c_1 + c_2 + c_3 = 3 \\ c_1 - c_2 + 2c_3 = -1 \\ c_1 \qquad + 2c_3 = 2 \end{cases}$$

We row reduce the corresponding augmented matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & -1 & 2 & -1 \\ 1 & 0 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Hence, $[q(t)]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix}$.
