Math 342 Solutions to Previous Midterm Exam

1. **True/False** If the statement is true, give a brief explanation; if it is false, provide a counterexample.

T **F** A subset H of a vector space V is a subspace if the zero vector is in H.

F A subspace must also be closed under addition and scalar multiplication. For example, $H = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ contains the zero vector. However, $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in H$, but $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \notin H$, so H is not a subspace.

T **F** The column space of A is the range of the mapping $\vec{x} \mapsto A\vec{x}$.

T The column space of A is the span of the columns of A. The range of $\vec{x} \mapsto A\vec{x}$ is the set of all multiples $A\vec{x}$, where $\vec{x}$ is a vector of the appropriate size. But the span of the columns of A is set of all linear combinations of the columns of A. By the definition of matrix multiplication, each $A\vec{x}$ is a linear combination of the columns of A, so the set of all such multiples is the set of all linear combinations of the columns of A.

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T **F** A single vector by itself is linearly dependent.

F In the first place, vectors are not linearly dependent or independent, *sets* of vectors are. Even if we read the question to say 'sets consisting of exactly one vector are dependent', the statement is false. For example, $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is a linearly independent set.

T **F** If there exists a linearly dependent set $\{\vec{v}_1, \ldots, \vec{v}_p\}$ in the vector space V, then $\dim V \leq p$.

F The subset $\{\vec{0}\} \subseteq \mathbb{R}^2$ is linearly dependent (*any* set containing the zero vector is dependent), but $\dim \mathbb{R}^2 = 2 \nleq 1$.

T **F** The dimension of the row space and the column space of A are the same, even if A is not square.

T This is one of the things the Rank Theorem says.

Problems

2. If the null space of a 7×6 matrix A is 5-dimensional, what is the dimension of the column space of A?

Solution: The Rank Theorem says $\operatorname{rk}(A) + \dim \operatorname{Nul}(A) =$ number of columns of A. Here we have $\operatorname{rk}(A) + 5 = 6$, so $\operatorname{rk}(A) = \dim \operatorname{Col}(A) = 1$.

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3. If A is a 7×5 matrix, what is the largest possible rank of A?

Solution: Since $\operatorname{rk}(A) + \dim \operatorname{Nul}(A) =$ number of columns of A, and all the number involved are nonnegative integers, the largest the rank can be is the number of columns of A. The largest rank a 7×5 matrix can have is 5.

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4. Let H be the set of all points in $\mathbb{R}^2$ with rational coordinates. That is, $H = \{(r,s) \mid r,s \in \mathbb{Q}\}$. Is H a subspace of $\mathbb{R}^2$? Explain.

Solution: No , H is not a subspace of $\mathbb{R}^2$. For example, $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in H$ and π is a scalar, but $\pi\vec{v} = \begin{bmatrix} \pi \\ 0 \end{bmatrix} \notin H$. Since it's not closed under scalar multiplication, H is not a subspace.

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5. Let F be a fixed 3×2 matrix and let H be the set of all 2×3 matrices A such that $AF = 0$ (the 2×2 zero matrix). Show that H is a subspace of $M_{2\times 3}$.

Solution: We must check three things:

a) H CONTAINS THE ZERO VECTOR. The zero of $M_{2\times3}$ is the 2 zero matrix—let's call it $O_{2\times3}$. But $O_{2\times3}F = O_{2\times2}$, so $O_{2\times3} \in H$

b) H IS CLOSED UNDER ADDITION. Let $A, B \in H$. Then $AF = O$ and $BF = O$. Hence, $(A+B)F = AF + BF = O + O = O$, so $A + B \in H$.

c) H IS CLOSED UNDER SCALAR MULTIPLICATION. Let $A \in H$ and let r be a scalar. Since $A \in H$, $AF = O$. But then $(rA)F = r(AF) = rO = O$, so $rA \in H$.

Therefore, H is indeed a subspace of $M_{2\times3}$.

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6. Let $T : M_{2\times2} \to M_{2\times2}$ be $T(A) = A - A^T$. This is a linear transformation. Describe the range and kernel of T.

Solution: We can actually write a formula for this transformation. An arbitrary element of $M_{2\times2}$ is $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where a, b, c, d are real numbers. Then

$$T(A) = A - A^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & b-c \\ c-b & 0 \end{bmatrix}.$$

This equals zero precisely when $b = c$. Notice also that $c - b = -(b - c)$. Hence,

$$\ker(T) = \{A \mid T(A) = O\} = \left\{ \begin{bmatrix} r & s \\ s & t \end{bmatrix} \ : \ r,s,t \in \mathbb{R} \right\}$$

$$\operatorname{range}(T) = \{T(A) \mid A \in M_{2\times2}\} = \left\{ \begin{bmatrix} 0 & z \\ -z & 0 \end{bmatrix} \ : \ z \in \mathbb{R} \right\}$$

We could also notice that $ker(T)$ is the set of symmetric matrices and range(T) is the set of skew-symmetric matrices (in $M_{2\times2}$).

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7. Find bases for Nul$A$, Col$A$, and Row$A$ if

$$A = \begin{bmatrix} 1 & -1 & 2 & 4 \\ -2 & 3 & -4 & -5 \\ 1 & -3 & 2 & -2 \end{bmatrix}$$

Solution: Of course, we row reduce.

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ -2 & 3 & -4 & -5 \\ 1 & -3 & 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & -2 & 0 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot columns of A are a basis for Col(A), so a basis for Col(A) is $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -3 \end{bmatrix} \right\}$.

The pivot rows of any echelon form of A are a basis for Row(A). Hence, a basis for Row(A) is $\{(1,0,2,7),(0,1,0,3)\}$.

The null space is the solution space to the homogeneous equation $A\vec{x} = \vec{0}$. From the reduced form of the matrix we see that

$$x_1 + 2x_3 + 7x_4 = 0$$

$$x_2 + 3x_4 = 0$$

Hence, $x_1 = -2x_3 - 7x_4$ and $x_2 = -3x_4$. Solutions look like

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_3 - 7x_4 \\ -3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

A basis for Nul(A) is $\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

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8. Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ and $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ be two bases for the vector space V, and suppose $\vec{b}_1 = 2\vec{c}_1 - 3\vec{c}_2$ and $\vec{b}_2 = 3\vec{c}_1 - 5\vec{c}_2$.

a) Find $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$

b) If $\vec{x} = 4\vec{b}_1 + 3\vec{b}_2$, find $[\vec{x}]_{\mathcal{C}}$.

c) If $\vec{y} = -7\vec{b}_1 + 4\vec{b}_2$, find $[\vec{y}]_{\mathcal{B}}$.

Solution: a) Well, $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} [\vec{b}_1]_{\mathcal{C}} & [\vec{b}_2]_{\mathcal{C}} \end{bmatrix}$, so $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 2 & 3 \\ -3 & -5 \end{bmatrix}$.

b) Well, $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$, and $[\vec{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\vec{x}]_{\mathcal{B}}$, so $[\vec{x}]_{\mathcal{C}} = \begin{bmatrix} 2 & 3 \\ -3 & -5 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 17 \\ -27 \end{bmatrix}$.

c) This is silly—we're told $\vec{y}$ in terms of $\mathcal{B}$, so we know $[\vec{y}]_{\mathcal{B}} = \begin{bmatrix} -7 \\ 4 \end{bmatrix}$.
