Math 342 Last Year's Final

1. **True/False** If the statement is true, give a brief explanation; if it is false, provide a counterexample.

T **F** $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.

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T **F** The set $H = \left\{ \begin{bmatrix} a-b \\ b \\ 2a+3b \end{bmatrix} : a, b \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^3$.

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Problems

2. If A is a 5×7 matrix and the rank of A is 4, what is $\dim \text{Nul}(A)$?

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3. Let $T : \mathbb{M}_{2\times 2} \to \mathbb{M}_{2\times 2}$ be the linear transformation defined by $T(A) = AS - SA$, where $S = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Describe the kernel and range of T.

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4. Let $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -6 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ -10 \\ 3 \\ 10 \end{bmatrix} \right\}$.

a) What is $\dim W$?

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b) Find a basis for W.

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5. Let $\mathcal{B} = \left\{ \vec{b}_1, \vec{b}_2 \right\} = \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ and $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\} = \left\{ \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 7 \\ -2 \end{bmatrix} \right\}$. Suppose that T is a linear transformation such that $T(\vec{b}_1) = 2\vec{b}_1 - \vec{b}_2$ and $T(\vec{b}_2) = -\vec{b}_1 + 3\vec{b}_2$. Find the matrix of T with respect to the basis $\mathcal{C}$.

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6. Find bases for Nul$A$, Col$A$, and Row$A$ if

$$A = \begin{bmatrix} 1 & -1 & 2 & 4 \\ -2 & 3 & -4 & -5 \\ 1 & -3 & 2 & -2 \end{bmatrix}$$

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7. Find the eigenvalues and corresponding eigenvectors of $A = \begin{bmatrix} 8 & -6 & 6 \\ -6 & 8 & -6 \\ -15 & 15 & -13 \end{bmatrix}$.

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8. Find the $\mathcal{B}$-matrix of the transformation $\vec{x} \mapsto A\vec{x}$, where $A = \begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix}$ and $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

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9. Show that for $n \times n$ matrices A, B and C, if A is similar to B and B is similar to C, then A is similar to C.

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10. Compute the characteristic polynomial and find the eigenvalues for $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & 2 \end{bmatrix}$.

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11. Let U be an orthogonal $n \times n$ matrix. Show that for any $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$, $(U\vec{x}) \bullet (U\vec{y}) = \vec{x} \bullet \vec{y}$.

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12. On $\mathbb{P}_2$, define an inner product by $\langle f , g \rangle = f(-1)g(-1) + f(0)g(0) + f(1)g(1)$. Compute:

a) $\| 1 \|$, $\| t \|$, $\langle 1 , t \rangle$, and $\langle t + 2 , t^2 - 1 \rangle$

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b) Use the Gram-Schmidt process to create an orthogonal basis from $\{1, t, t^2\}$.

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